Czechoslovak Mathematical Journal

Charles W. Swartz

Weak Fubini theorems for the Dobrakov integral

Czechoslovak Mathematical Journal, Vol. 30 (1980), No. 4, 647–654

Persistent URL: http://dml.cz/dmlcz/101713

Czechoslovak Mathematical Journal, 30 (105) 1980, Praha

WEAK FUBINI THEOREMS FOR THE DOBRAKOV INTEGRAL

CHARLES SWARTZ, Las Cruces

(Received January 29, 1979)

In [6], I. DOBRAKOV has given a very thorough and interesting treatment of the Fubini Theorem for the integration of vector-valued functions with respect to operator-valued measures. This note contains some additional observations on such Fubini-type theorems. In particular we establish several weak-type Fubini Theorems where the functions under consideration may only be scalarly integrable and not necessarily integrable. Examples are presented which show that this type of phenomena can actually occur.

Throughout let $\mathcal{M}$, $\mathcal{N}$ be σ-algebras of subsets of S, T, respectively. Let X, Y, Z be (real) B-spaces with $L(X, Y)$ the space of bounded linear operators from X into Y. Let $\mu : \mathcal{M} \to L(X, Y)$, $\nu : \mathcal{N} \to L(Y, Z)$ be set functions which are of finite semi-variation and countably additive in the strong operator topology. We can define a product measure, $\mu \times \nu$, on measurable rectangles $A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$, by $\mu \times \nu(A \times B) = \nu(B)\,\mu(A)$. The product has a finitely additive extension to the algebra $\mathcal{A}$ generated by the measurable rectangles and we say that the product $\mu \times \nu$ exists if $\mu \times \nu$ has an extension to the σ-algebra Σ generated by $\mathcal{A}$ which is countably additive with respect to the strong operatory topology on $L(X, Z)$ ([6], Def. 1.).

An $\mathcal{N}$-measurable function $h : T \to Y$ is said to be *scalarly ν-integrable* if h is $z'\nu$-integrable for each $z' \in Z'$. (Here, $z'\nu$ is the $Y' = L(Y, \mathbb{R})$-valued measure $z'\,\nu(B)\,(y) = \langle z', \nu(B)\,y \rangle$.) If h is scalarly ν-integrable, its ν-integral over any $B \in \mathcal{N}$, $\int_B h \,\mathrm{d}\nu$, is the element of Z'' defined by $\langle \int_B h \,\mathrm{d}\nu, z' \rangle = \int_B h \,\mathrm{d}z'\nu$ (see [10] for the properties of scalarly integrable functions and the connection with the Dobrakov integral developed in [4], [5].).

In Theorem 2 we present a result analogous to Theorem 15 of [6] for scalarly integrable functions. In Examples 3 and 4 we present examples which show that the Fubini Theorem may indeed hold for functions which have integrals which are Z''-valued. This result thus represents an interesting appendix to the general Fubini Theorem of Dobrakov ([6], Th. 15). In Example 5, we present an example which illustrates how pathological the "partial integral" may be when the sections of the function are only assumed to be scalarly integrable.

As noted by Dobrakov in Example 1 of [6], in trying to establish a Fubini-type result for a function $f : S \times T \to X$ it is necessary to assume that the sections $f(\cdot, t)$ are integrable. This condition can be difficult to establish in practice and it would be desirable to have some sort of condition on the function f which would insure this as well as the measurability of the "partial integral" $t \to \int_S f(\cdot, t)\,\mathrm{d}\mu$. That is, it would be desirable to have a Tonelli-type of result for the Dobrakov integral. Such results have been derived for the more restrictive integral of Dinculeanu ([3]) in [1] and [8]. In Theorem 8 we present such a Tonelli-type result and compare it to the Theorems of Huneycutt ([8]) and Bandyopadhyay ([1]).

We first establish a technical lemma which will contain results which are used later. For any measure ν its variation will be denoted by $|\nu|$ ([3], § 3). If $z' \in Z'$, then $z'\nu$ is a $Y' = L(Y, \mathbb{R})$-valued measure so the product $\mu \times z'\nu$ can be formed resulting in an $X' = L(X, \mathbb{R})$-valued measure. The lemma contains some statements concerning this product.

Lemma 1. *Let $z' \in Z'$, $h : T \to Y$ be $\mathcal{N}$-measurable and $\mu \times \nu$ exist.*

(i) $z'(\mu \times \nu) = \mu \times z'\nu$,
(ii) $|z'\mu \times \nu| \leqq |\mu| \times |z'\nu| \leqq \|z'\|\ |\mu| \times |\nu|$,
(iii) *if* $\int_T \|h\|\ \mathrm{d}|z'\nu| < \infty$ *for each* z',

then h is scalarly ν-integrable.

Proof. For (i) if $A \times B$ is a measurable rectangle, then $\langle z', \mu \times \nu(A \times B)\rangle = z'\,\nu(B)\,\mu(A)$ so (i) holds. The product $z'(\mu \times \nu)$ exists by the assumption that the product $\mu \times \nu$ exists and the product $\mu \times z'\nu$ exists since $z'\nu$ is of bounded variation ([6] Th. 3; [9] Cor. 1). Hence, (i) must hold on Σ by uniqueness.

The first inequality in (ii) holds by Theorem 1(iv) of [1]. The second inequality in (ii) holds since $|z'\nu| \leqq \|z'\|\ |\nu|$.

For (iii), Theorem 6 of [4], implies that h is $z'\nu$-integrable for each $z' \in Z'$, i.e., h is scalarly ν-integrable.

We now establish the weak Fubini Theorem. The result is the direct analogue of Theorem 15 of [6] for scalarly integrable functions.

Theorem 2. *Let $f : S \times T \to X$ be Σ-measurable with $f(\cdot, t)$ μ-integrable for each t and $F : t \to \int_S f(\cdot, t)\,\mathrm{d}\mu$ ν-measurable. Then f is scalarly $\mu \times \nu$-integrable iff F is scalarly ν-integrable and in this case*

$$\text{(F)} \qquad \int_{S\times T} f\,\mathrm{d}\mu \times \nu = \int_T F\,\mathrm{d}\nu = \int_T \int_S f(s, t)\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t)\,,$$

where the integrals may be Z''-valued.

Proof. Let $z' \in Z'$. If f is scalarly $\mu \times \nu$-integrable from Dobrakov's Fubini Theorem ([6], Th. 15) applied to the measure $z'(\mu \times \nu) = \mu \times z'\nu$ (Lemma 1 (i)), we have $\int_{S\times T} f\, dz'(\mu \times \nu) = \int_T \int_S f(s, t)\, d\mu(s)\, dz'\, \nu(t)$. Hence F is scalarly ν-integrable and (F) holds.

The converse is established in exactly the same way.

For bounded functions we have the following general result (compare with Theorem 16 of [6] and Theorem 8 of [7]).

Corollary 3. *Let f be bounded and satisfy the hypothesis of Theorem* 2. *Then* (F) *holds.*

Proof. Note the function F is bounded ([4], Th. 14) so both f and F are scalarly integrable ([4], Th. 6).

A measure $\mu : \mathscr{M} \to L(X, Y)$ is said to be *strongly bounded* (dominated or with continuous semi-variation) if whenever $\{A_n\}$ is a sequence from $\mathscr{M}$ which decreases to the null set, the semi-variation, $\hat{\mu}$, of μ satisfies $\hat{\mu}(A_n) \to 0$. Recall, if μ is strongly bounded and $h : S \to X$ is bounded and $\mathscr{M}$-measurable, then h is μ-integrable ([4], Th. 5). For strongly bounded measures, we have the following:

Remark 4. If the measure μ is strongly bounded, each $f(\cdot, t)$ in Corollary 3 is μ-integrable ([4], Th. 5). Moreover, if $\{f_n\}$ is a sequence of simple functions which converge pointwise to f on $S \times T$ with $\|f_n(s, t)\| \leqq \|f(s, t)\|$ for $s \in S$, $t \in T$, then the Bounded Convergence Theorem for μ gives $\int_S f_n(\cdot, t)\, d\mu \to \int_S f(\cdot, t)\, d\mu$ so that F is measurable and (F) always holds in this case. If ν is also strongly bounded, then f is also $\mu \times \nu$-integrable since $\mu \times \nu$ is then strongly bounded ([7], Th. 6) and (F) holds with the integrals being Z-valued. This gives the analogue of Theorem 8 of [7] for the Dobrakov integral. Since the Dobrakov integral is more general than the Bartle integral treated in [7], this generalizes Duchoň-'s result.

We now present two examples which show that the integrals in (F) of Theorem 2 can indeed be Z''-valued, i.e., a weak-type Fubini Theorem holds. The first example is extremely simple and is presented for that reason. The measures involved are only countably additive in the strong operator topology and not in the uniform operator topology. The second example shows that equation (F) may hold in Z'' even when the measure μ is countably additive in the uniform operator topology. It is somewhat more complex than Example 5 and is modelled on some of the examples of [4].

Example 5. Let e_n be the n^{th} unit vector, $e_n = \{e_{nj}\}_{j=1}^{\infty}$. Define $f : \mathbb{N} \times \mathbb{N} \to c_0$ by $f(k, j) = \delta_{kj} e_j$. For each k define $\mu_k \in L(c_0, c_0)$ by $\mu_k x = x_k e_k$, where $x = \{x_j\}$. The sequence $\{\mu_k\}$ induces a measure on the power set, $\mathscr{P}$, of $\mathbb{N}$ by $\mu(E) = \sum_{k \in E} \mu_k$ since the series $\sum_{k=1}^{\infty} \mu_k$ is unconditionally convergent in the strong operator topology. Thus, the measure μ is countably additive in the strong operator topology and has finite semi-variation with $\hat{\mu}(E) = 1$ for E non-void. Since $\|\mu(E)\| = 1$, μ is, however, not countably additive in the uniform operator topology.

The product measure exists since $\mu \times \mu(i, j)\, x = \delta_{ij} x_j e_i$ and the series $\sum_{i,j} \mu \times \times \mu(i, j)\, x$ is unconditionally convergent in c_0 for each $x \in c_0$. Actually, the measure $\mu \times \mu$ is supported on the diagonal of $\mathbb{N} \times \mathbb{N}$ and has essentially the same form as μ. In particular, $\mu \times \mu$ has bounded semi-variation and is countably additive in the strong operator topology.

First we have $\int_{\mathbb{N} \times \mathbb{N}} f \,\mathrm{d}\mu \times \mu = \sum_{i=1}^{\infty} \mu \times \mu(i, i) f(i, i) = \sum_{i=1}^{\infty} e_i$, the series being weak* convergent in ℓ^{∞}. The "partial integral" is $\int_{\mathbb{N}} f(k, j)\,\mathrm{d}\mu(k) = F(j) = \sum_{k=1}^{\infty} \mu_k \delta_{kj} e_j = e_j$ so each $f(\cdot, j)$ is μ-integrable, and we have $\int_{\mathbb{N}} F\,\mathrm{d}\mu = \sum_{j=1}^{\infty} \mu_j e_j = \sum_{j=1}^{\infty} e_j$.

Thus, the integrals in (F) may indeed be scalar-type integrals with values in Z'' and not in Z.

Example 6. Partition $\mathbb{N}$ by $\sigma_j = \{2^{j-1}, 2^{j-1} + 1, \ldots, 2^j - 1\}$, $j = 1, 2, \ldots$, and note each σ_j consists of 2^{j-1} integers. For each k define $y_k = e_j/2^{j-1}$ if $k \in \sigma_j$ and define $\mu_k \in L(\ell^1. c_0)$ by $\mu_k x = x_k y_k$. Then $\{\mu_k\}$ induces a measure on $\mathscr{P}$ to $L(\ell^1, c_0)$ by $\mu(E) = \sum_{k \in E} \mu_k$, where the series converges unconditionally in the strong operator topology since $\sum_{k=1}^{\infty} \mu_k x = \sum_{j=1}^{\infty} \sum_{k \in \sigma_j} x_k e_j/2^{j-1}$ and $\{\sum_{k \in \sigma_j} x_k\}_j$ is bounded for $x = \{x_k\} \in \ell^1$. Actually the measure μ is countably additive in the uniform topology since it is countably additive in the strong operator topology and $\|\mu_k\| = \|y_k\| \to 0$. Since $\hat{\mu}(E) = 1$ for non-void E, μ has bounded semi-variation.

Take v to be the measure of Example 5. Then the product $\mu \times v$ exists since $\mu \times v(k, i)\, x = v_i(\mu_k x) = x_k \delta_{ij} e_i/2^{j-1}$ for $k \in \sigma_j$ implies the series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \mu \times v(k, i)\, x = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Big(\sum_{k \in \sigma_j} x_k\Big) \delta_{ij} e_i/2^{j-1} = \sum_{i=1}^{\infty} \Big(\sum_{k \in \sigma_i} x_k\Big) e_i/2^{i-1}$$

is unconditionally convergent in c_0 for $x \in \ell^1$.

Define $f : \mathbb{N} \times \mathbb{N} \to \ell^1$ by $f(k, i) = e_k$ if $k \in \sigma_i$ and $f(k, i) = 0$ otherwise. First we have $\int_{\mathbb{N} \times \mathbb{N}} f\,\mathrm{d}\mu \times v = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k \in \sigma_j} \delta_{ij} e_i/2^{j-1} = \sum_{i=1}^{\infty} e_i$. For the "partial integral", $F(i) = \int_{\mathbb{N}} f(k, i)\,\mathrm{d}\mu(k) = \sum_{k \in \sigma_i} \mu_k e_k = e_i$, so $f(\cdot, i)$ is μ-integrable and $\int_{\mathbb{N}} F\,\mathrm{d}v = \sum_{i=1}^{\infty} v_i e_i = \sum_{i=1}^{\infty} e_i$.

One of the annoying features of Theorem 2 and Theorem 15 of [6] is the assumption of measurability of the "partial integral", $t \to \int_S f(\cdot, t)\,\mathrm{d}\mu$. We next present an example which shows that if the sections $f(\cdot, t)$ are assumed only to be scalarly integrable then the partial integral may indeed fail to be measurable. In light of the results of [6], it would be desirable to know if the same phenomena can occur when the sections $f(\cdot, t)$ are actually integrable.

Example 7. Let $A_1 = [0, 1]$, $A_2 = [0, 1/2]$, $A_3 = [1/2, 1]$, $A_4 = [0, 1/4], \dots$ as in Example II.1.7 of [2]. Define $f : \mathbb{N} \times [0, 1] \to c_0$ by $f(k, t) = C_{A_k}(t)\, e_k$. Note f is measurable since each coordinate function of f is measurable (II.1.4 of [2]). Let μ be the measure of Example 5, $\mu : \mathcal{P} \to L(c_0, c_0)$. The function $f(\cdot, t)$ is bounded and, therefore, scalarly μ-integrable with

$$F(t) = \int_{\mathbb{N}} f(k, t)\, \mathrm{d}\mu(k) = \sum_{k=1}^{\infty} \mu_k f(k, t) = \sum_{k=1}^{\infty} C_{A_k}(t)\, e_k\,,$$

the series converging weak* in ℓ^∞. But as shown in Example II.1.7 of [2] the function F is not measurable with respect to Lebesgue measure on $[0, 1]$.

Example 7 does show that if one is dealing with measures which are countably additive in a topology weaker than the strong operator topology, then the partial integral may fail to be measurable. This suggests that it may be difficult to formulate very satisfactory Fubini-type theorems in locally convex spaces. These remarks follow by noting that in Example 7 the function f is ℓ^∞-valued and the measure μ can be considered to have values in $L(\ell^\infty, \ell^\infty)$. The measure μ is thencountab ly additive with respect to the topology of pointwise convergence on ℓ^∞ when ℓ^∞ has the weak* topology. The functions $f(\cdot, t)$ are then "weakly μ-integrable" (in the sense of [10]) with $F(t) = \int_{\mathbb{N}} f(\cdot, t)\, \mathrm{d}\mu = \sum_k C_{A_k}(t)\, e_k$, the series being weak* convergent.

One of the difficulties in dealing with Fubini-type results for products of vector-valued measures is that a function $f : S \times T \to X$ may be badly behaved on sets where the product measure $\mu \times \nu$ is zero due to the presence of zero-divisors. This means that f may be integrable with respect to $\mu \times \nu$ while the sections $f(\cdot, t)$ are badly behaved over sets with non-zero μ measure. This is essentially the thrust of Example 1 of [6]. In our next result we present a Tonelli-type result which gives conditions for the sections $f(\cdot, t)$ to be well-behaved. To accomplish this we deal with the measures $|\mu| \times |z'\nu|$ in place of the measures $z'\mu \times \nu$ of Theorem 2. Dealing with these measures overcomes the difficulties discussed above.

Theorem 8. *Let μ have σ-finite variation and suppose there is $z_0' \in Z'$ such that $|z_0'\nu|(B) = 0$ implies $\hat{\nu}(B) = 0$. If f is Σ-measurable and $\int_{S \times T} \|f(s, t)\|\, \mathrm{d}|\mu| \times |z'\nu|(s, t) < \infty$ for each $z' \in Z'$, then $f(\cdot, t)$ is μ-integrable for $\hat{\nu}$-almost all t, $F : t \to \int_S f(\cdot, t)\, \mathrm{d}\mu$ is $\hat{\nu}$-measurable, f is scalarly $\mu \times \nu$-integrable and* (F) *holds.*

Proof. By the classical scalar Tonelli Theorem,

$$\int_T \int_S \|f(s, t)\|\, \mathrm{d}|\mu|(s)\, \mathrm{d}|z_0'\nu|(t) = \int_{S \times T} \|f\|\, \mathrm{d}|\mu| \times |z_0'\nu| < \infty\,.$$

Hence $\int_S \|f(s,t)\| \, d|\mu| (s) < \infty$ for $|z_0'\nu|$-almost all t and, hence, for $\hat{\nu}$-almost all t. By [4], Theorem 6, $f(\cdot, t)$ is μ-integrable and by [6], Theorem 14, the function $F : t \to \int_S f(\cdot, t) \, d\mu$ is $\hat{\nu}$-measurable. Since

$$\int_T \|F(t)\| \, d|z'\nu| (t) \leqq \int_T \int_S \|f(s,t)\| \, d|\mu| (s) \, d|z'\nu| (t) < \infty \quad \text{for} \quad z' \in Z' ,$$

Lemma 1 (iii) yields that F is scalarly ν-integrable. Theorem 2 gives the desired conclusion.

Note that Example 5 again illustrates that the integrals in (F) may indeed by Z''-valued even when the hypothesis of Theorem 8 are satisfied.

The conditions of Theorem 8 are only sufficient conditions for the conclusions of the Fubini Theorem to hold; they are by no means necessary as the following example shows.

Example 9. Let $\{t_j\}$ be a sequence in c_0 but not in ℓ^1. Define $f : \mathbb{N} \times \mathbb{N} \to c_0$ by $f(k, j) = t_j e_j$. Let μ be the measure of Example 5. Then $\int_{\mathbb{N}} f(k, j) \, d\mu(k) = t_j e_j = F(j)$ and $\int_{\mathbb{N} \times \mathbb{N}} f \, d\mu \times \nu = \int_{\mathbb{N}} F \, d\mu = \sum_j t_j e_j$. Thus, both f and F are integrable. However, if $z' = \{s_j\} \in \ell^1$ is non-zero. then $\int_{\mathbb{N} \times \mathbb{N}} \|f\| \, d|\mu| \times |z'\mu| = \infty$ and the condition of Theorem 8 does not hold.

In attempting to apply Theorem 8 it is desirable to have conditions which insure that the hypothesis on the measure ν holds. We next observe that the condition on ν always holds if ν is strongly bounded and, thus, Theorem 8 always holds for strongly bounded ν.

Lemma 10. *Let ν be strongly bounded and set $K = \{|z'\nu| : \|z'\| \leqq 1\}$. Then there is a $z_0' \in Z'$, $\|z_0'\| \leqq 1$, such that K is uniformly absolutely continuous with respect to $|z_0'\nu|$.*

Proof. K is a bounded convex subset of $ca(\mathscr{N})$ which is uniformly countably additive by the strong boundedness of ν. The conclusion follows from [2], IX.2.5.

The strong boundedness of ν is a sufficient condition for the hypothesis of Theorem 8 to hold. The measure μ of Example 5 shows that it is not necessary. (For example, take $z_0' = \{1/n^2\}$.)

Concerning the integrability of the function f in Theorem 8 we have the following:

Corollary 11. *Let μ, ν be as in Theorem 8. If the set function $\alpha : E \to \sup_{\|z'\| \leqq 1} \int_E \|f(s,t)\| \, d|\mu| \times |z'\nu| (s,t)$ $(E \in \Sigma)$ is continuous from above at $\emptyset$, then the conclusions of Theorem 8 hold with f $\mu \times \nu$-integrable.*

Proof. From Lemma 1 (ii), $|z'\mu \times \nu| \leqq |\mu| \times |z'\nu|$. Therefore, α being continuous at $\emptyset$ implies that the function f is in $L_1(\mu \times \nu)$ ([5], Def. 4) and is hence $\mu \times \nu$-integrable ([5], Lemma 1).

Huneycutt ([8]) and Bandyopadhyay ([1]) have established Fubini-Tonelli Theorems for the more restrictive integration theory of N. DINCULEANU ([3]). Due to the fact that the Dobrakov integral is so much more general than the integral of Dinculeanu, we can derive the results Huneycutt and Bandyopadhyay from Corollary 11.

Corollary 12. *Let μ and ν have σ-finite variation. Let $f : S \times T \to X$ be Σ-measurable and satisfy $\int_{S\times T} \|f(s, t)\| \, \mathrm{d}|\mu| \times |\nu| (s, t) < \infty$. Then the conclusions of Theorem 8 hold with f being $\mu \times \nu$-integrable.*

Proof. From Lemma 1 (ii), $|\mu| \times |z'\nu| \leqq \|z'\| \, |\mu| \times |\nu|$. Hence, the hypothesis of Corollary 11 are satisfied since $E \to \int_E \|f\| \, \mathrm{d}|\mu| \times |\nu|$ is countably additive.

The following example illustrates that Theorem 8 is more general than Corollary 12. This is basically due to the fact that the Dobrakov integral is more general than ιhen integral of Dinculeanu.

Example 13. Let the sequence $x = \{t_j\}$ belong to c_0 but not to ℓ^1. Define $f : \mathbb{N} \times \mathbb{N} \to c_0$ by $f(k, j) = \delta_{kj} t_j e_j$. Let μ be the measure of Example 5. Then $\int_{\mathbb{N}\times\mathbb{N}} \|f\| \, \mathrm{d}|\mu| \times |\mu| = \sum_j |t_j| = \infty$ so the condition of Corollary 12 does not hold. Let D be the diagonal in $\mathbb{N} \times \mathbb{N}$, $D = \{(n, n) : n \in \mathbb{N}\}$. If $z' = \{s_j\} \in \ell^1$, $\int_{\mathbb{N}\times\mathbb{N}} \|f\| \, . \, \mathrm{d}|\mu| \times |z'\mu| = \sum_k |t_k s_k| < \infty$ so Theorem 8 is applicable. In this example f is actually integrable with $\int_{\mathbb{N}\times\mathbb{N}} f \, \mathrm{d}\mu \times \mu = \sum_{j=1}^{\infty} t_j e_j$. Actually Corollary 11 is applicable here since if D is the diagonal of $\mathbb{N} \times \mathbb{N}$, $D = \{(n, n) : n \in \mathbb{N}\}$,

$$\sup_{\|z'\| \leqq 1} \int_E \|f\| \, \mathrm{d}|\mu| \times |z'\mu| = \sup_{\|z'\| \leqq 1} \sum_{k \in E \cap D} |s_k t_k| \leqq \sup_{k \in E \cap D} |t_k| \, .$$

This shows that Corollary 11 gives a more general Tonelli-type result than Theorem 2 of [1].

References

[1] *U. K. Bandyopadhyay:* On products of vector measures. J. Australian Math. Soc., *19* (1975), 91—96.
[2] *J. Diestel* and *J. J. Uhl:* Vector measures, Amer. Math. Soc., Math. Surveys 15, 1977.
[3] *N. Dinculeanu:* Vector measures, Pergamon, New York, 1967.
[4] *I. Dobrakov:* On integration in Banach spaces I, Czech. Math. J., *20* (1970), 511—536.
[5] *I. Dobrakov:* On integration in Banach spaces II, Czech. Math. J., *20* (1970), 680—695.
[6] *I. Dobrakov:* On integration in Banach spaces III, Czech. Math. J., *29* (1979), 478—499.
[7] *M. Duchoň:* Product of dominated vector measures, Math. Slovaca, *27* (1977), 293—301.
[8] *J. Huneycutt:* Products and convolutions of vector valued set functions, Studia Math. *41* (1972), 119—129.

[9] *C. Swartz:* A generalization of a Theorem of Duchoň on products of vector measures, J. Math. Anal. Appl. *51* (1975), 621–628.
[10] *C. Swartz:* Integrability for the Dobrakov integral, preprint.

Author's address: New Mexico State University, College of Arts and Sciences, Las Cruces, New Mexico 88003, U.S.A.