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# THE BICHROMATICITY OF A GRAPH

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F. HARARY, D. HSU and Z. MILLER in [1] define the bichromaticity of a bipartite graph as follows. Let $B$ be a connected finite bipartite graph on the vertex sets $U$, $V$. (This means that the vertex set of $B$ is the union of disjoint sets $U$, $V$ and each edge of $B$ joins a vertex of $U$ with a vertex of $V$.) A bicomplete homomorphism of the graph $B$ onto a complete bigraph $K_{r,s}$ is such a homomorphism $\psi$ of $B$ onto $K_{r,s}$ that for any two vertices $x$, $y$ of $B$ the equality $\psi(x) = \psi(y)$ implies that $x$ and $y$ are either both in $U$ or both in $V$. The bichromaticity $\beta(B)$ of the graph $B$ is the maximal number of vertices of a complete bipartite graph onto which $B$ can be mapped by a bicomplete homomorphism.

Following [1], a bipartite graph will be shortly called a bigraph. If $B$ is a bigraph on the sets $U$, $V$ and $|U| \geqq |V|$, then $U$ is called the majority of $B$ and its cardinality is denoted by $\mu(B)$.

In [1] a problem was proposed to find $\beta(B \times K_2)$ in terms of $\beta(B)$. The authors noted that an exact formula determining $\beta(B \times K_2)$ as a function of $\beta(B)$ would yield a formula for $\beta(Q_n)$, where $Q_n$ is the graph of the cube of dimension $n$. We shall show that $\beta(B \times K_2)$ is not uniquely determined by $\beta(B)$ and we shall give bounds for $\beta(B \times K_2)$ in terms of $\beta(B)$. First we prove a theorem which will serve us as a lemma.

**Theorem 1.** *Let $B$ be a connected finite bigraph on the sets $U$, $V$. Let there exist a vertex $x \in U$ which is adjacent to all vertices of $V$ and let the bigraph $B_0$ obtained from $B$ by deleting $u$ be connected. Then*

$$\beta(B) = \beta(B_0) + 1 .$$

Proof. Let $B_0$ be mapped by a bicomplete homomorphism $\psi_0$ onto the complete bigraph $K_{r,s}$ such that $r + s = \beta(B_0)$. Let $U - \{x\}$ (or $V$) be mapped by this homomorphism onto a set $U_0$ (or $V_0$) of the cardinality $r$ (or $s$, respectively). Consider the graph $K_{r+1,s}$ obtained from $K_{r,s}$ by adding a vertex $y$ and joining it by edges with all vertices of $V_0$. Then we define the mapping $\psi$ of $B$ onto $K_{r+1,s}$ such that $\psi(v) = \psi_0(v)$ for each $v \neq x$ and $\psi(x) = y$. This is evidently a bicomplete homomorphism of $B$ onto $K_{r+1,s}$ and $\beta(B) \geqq \beta(B_0) + 1$. Now suppose $\beta(B) \geqq \beta(B_0) + 2$. Then there exists a bicomplete homomorphism $\psi'$ of $B$ onto a bigraph $K_{p,q}$ such that $p + q \geqq r + s + 2$. Let $U$ (or $V$) be mapped by $\psi'$ onto a set of the cardinality $p$ (or $q$, respectively). By deleting $\psi'(x)$ from $K_{p,q}$ we obtain the graph $K_{p-1,q}$. The mapping $\psi'$ maps an induced subgraph of $B_0$ onto this graph $K_{p-1,q}$, therefore also the whole graph $B_0$ can be mapped by a bicomplete homomorphism onto $K_{p-1,q}$. But $p - 1 + q \geqq r + s + 1$, which is a contradiction with the assumption that $\beta(B_0) = r + s$. Therefore the assertion of the theorem is true.

Now we prove a theorem on $\beta(B \times K_2)$.

**Theorem 2.** *Let $B$ be a connected finite bigraph which can be mapped by a bicomplete homomorphism onto the complete bigraph $K_{r,s}$, where $r \leqq s$. Then $B \times K_2$ can be mapped by a bicomplete homomorphism onto the complete bigraph $K_{r,r+s}$. There exist bigraphs $B$ with the property that $B$ can be mapped by a bicomplete homomorphism onto $K_{r,s}$, where $r \leqq s$ and $\beta(B \times K_2) = 2r + s$.*

Proof. Let $B$ be a bigraph on the sets $U$, $V$. The graph $B \times K_2$ can be described as follows: Take two copies $B'$ and $B''$ of the graph $B$ and an isomorphic mapping $\varphi$ of $B'$ onto $B''$ and join each vertex $x$ of $B'$ with its image $\varphi(x)$ by an edge. The sets corresponding to $U$, $V$ in $B'$ will be denoted by $U'$, $V'$ and in $B''$ by $U''$, $V''$. Without loss of generality suppose that in the bicomplete homomorphism $\psi$ which maps $B'$ onto $K_{r,s}$ the set $U$ is mapped onto a set $U_0$ of the cardinality $r$ and $V$ is mapped onto a set $V_0$ of the cardinality $s$. Let $K', K''$ be two copies of $K_{r,s}$. We map $B'$ onto $K'$ and $B''$ onto $K''$ by a bicomplete homomorphism corresponding to $\psi$. The images of $U', U'', V', V''$ will be consequently $U'_0, U''_0, V'_0, V''_0$; we have $|U'_0| = |U''_0| = r$, $|V'_0| = |V''_0| = s$. Now we choose a surjection $\varphi : V''_0 \to U'_0$ and we identify each $x \in U''_0$ with its image $\varphi(x)$. Thus we obtain a complete bigraph $K_{r,r+s}$ onto which $B \times K_2$ is mapped.

Now let $B$ be a complete bigraph $K_{2,s}$, where $s > 2$. We shall prove that $\beta(K_{2,s} \times K_2) = s + 4$. We use the notation introduced above; we have $|U| = |U'| = |U''| = 2$, $|V| = |V'| = |V''| = s$. Let $K_{p,q}$ be a complete bigraph onto which $K_{2,s} \times K_2$ can be mapped by a bicomplete homomorphism and such that $p + q = \beta(K_{2,s} \times K_2)$. If each vertex $x \in U' \cup U''$ has the property that $\psi(x) \neq \psi(y)$ for each $y \neq x$, where $\psi$ is the bicomplete homomorphism of $K_{2,s} \times K_2$ onto $K_{p,q}$, then neither the subgraph of $K_{p,q}$ induced by $\psi(U' \cup U'')$ nor $K_{p,q}$ are complete bigraphs which is a contradiction. Therefore there exists at least one vertex $x \in U' \cup U''$ to which a vertex $y$ exists such that $x \neq y$ and $\psi(x) = \psi(y)$. Without loss of generality let $x \in U'$. Then $y \in U' \cup V''$. Let $G_0$ be the graph obtained from $K_{2,s}$ by identifying $x$ and $y$; the graph $G_0$ can be mapped by a bichromatic homomorphism onto $K_{p,q}$ and $\beta(G_0) = \beta(K_{2,s})$. If $y \in U'$, then the vertex obtained by identifying $x$

and $y$ is adjacent to all vertices of $U'' \cup V'$ in $G_0$. Let $G_1$ be the bigraph obtained from $G_0$ by deleting this vertex; by Theorem 1 we have $\beta(G_1) = \beta(G_0) - 1$. Let $z \in U''$. The graph $G_1$ is a bigraph on the set $V''$, $U'' \cup V'$ and $z$ is adjacent to all vertices of $V''$ in $G_1$. Let $G_2$ be the bigraph obtained from $G_1$ by deleting $z$; we have $\beta(G_2) = \beta(G_1) - 1 = \beta(G_0) - 2$. But $G_2$ is a tree with the majority $(U'' - \{z\}) \cup V'$ of the cardinality $\mu(G_2) = s + 1$, therefore by Theorem 1 from [1] we have $\beta(G_2) = s + 2$ and this implies $\beta(G) = \beta(G_0) = s + 4$. Now suppose that the images of vertices of $U'$ in $\psi$ are different. If the images of vertices of $U''$ are equal, then we proceed analogously as in the preceding case. Therefore suppose also that the images of vertices of $U''$ in $\psi$ are different. Let $U' = \{u'_1, u'_2\}$, $U'' = \{u''_1, u''_2\}$ and let $u'_1$ be adjacent with $u''_1$ and $u'_2$ with $u''_2$ in $K_{2,s} \times K_2$. Then one of the following four cases must occur:

(a) There exist vertices $y_1, y_2$ of $V''$ such that $y_1 \neq y_2$ and $\psi(u'_1) = \psi(y_1)$, $\psi(u'_2) = \psi(y_2)$.

(a′) There exist vertices $z_1, z_2$ of $V'$ such that $z_1 \neq z_2$ and $\psi(u''_1) = \psi(z_1)$, $\psi(u''_2) = \psi(z_2)$.

(b) There exist vertices $y_1 \in V''$, $z_2 \in V'$ such that $\psi(u'_1) = \psi(y_1)$, $\psi(u''_2) = \psi(z_2)$.

(b′) There exist vertices $y_2 \in V''$, $z_2 \in V'$ such that $\psi(u'_2) = \psi(y_2)$, $\psi(u''_2) = \psi(z_2)$.

If the case (a) occurs, let $G_0$ be the graph obtained from $K_{2,s} \times K_2$ by identifying $u'_1$ with $y_1$ and $u'_2$ with $y_2$. We have then $\beta(G_0) = p + q$. The graph $G_0$ is a bigraph on the sets $V''$, $U'' \cup V'$ and the vertices $u''_1, u''_2$ are adjacent with all vertices of $V''$ in $G_0$. Let $G_1$ be the graph obtained from $G_0$ by deleting $u''_1$ and $u''_2$; by Theorem 1 we have $\beta(G_1) = \beta(G_0) - 2$. Each vertex $V'' - \{y_1, y_2\}$ has degree 1 in $G_1$ and the subgraph of $G_1$ induced by $\{y_1, y_2\} \cup U'' \cup V'$ is isomorphic to $K_{2,s}$. Evidently, if $G_1$ is mapped onto a complete bigraph by a bicomplete homomorphism, then either all vertices of $U'' \cup V'$ are mapped onto the same vertex, or each vertex of $V'' - \{y_1, y_2\}$ is mapped onto the same vertex as $u'_1$ or $u'_2$. In the first case the mentioned complete bigraph is $K_{1,s}$, in the second case $K_{2,s}$. Therefore $\beta(G_1) = s + 2$ and $\beta(G_0) = \beta(K_{2,s} \times K_2) = s + 4$. The case (a′) is analogous. Let the case (b) occur and let $G_0$ be the bigraph obtained from $K_{2,s} \times K_2$ by identifying $u'_1$ with $y_1$ and $u''_1$ with $z_1$. The vertices $u'_2, u''_2$ fulfil the condition of Theorem 1; let $G_1$ be the graph obtained from $G_0$ by deleting them. We have $\beta(G_1) = \beta(G_0) - 2$. The graph $G_1$ is a tree with the majority of the cardinality $s$, therefore $\beta(G_1) = s + 1$ and $\beta(K_{2,s} \times K_2) = \beta(G_0) = s + 3$. This is impossible, since by the above proved results $\beta(K_{2,s} \times K_2) \geqq s + 4$. Analogously in the case (b′). Therefore one of the cases (a), (a′) occurs and $\beta(K_{2,s} \times K_2) = s + 4$. As $r = 2$, this is $2r + s$.

**Theorem 3.** *Let $B$ be a connected finite bigraph on the sets $U$, $V$ which can be mapped by a bicomplete homomorphism onto a complete bigraph $K_{r,s}$, where $r \leqq s$. Let there exist an automorphism of $B$ which maps $U$ onto $V$ and $V$ onto $U$. Then $B \times K_2$ can be mapped by a bicomplete homomorphism onto a complete bigraph $K_{r,2s}$.*

Proof. We use the same notation as in the proof of Theorem 2. Note that if $U$ can be mapped onto $U_0$ and $V$ onto $V_0$ by a bicomplete homomorphism, then also $U$ can be mapped onto $V_0$ and $V$ onto $U_0$ (superposing the original homomorphism with the mentioned automorphism). Then also the images of $U', U'', V', V''$ can be consequently $U_0', V_0'', V_0', U_0''$. By an analogous procedure as in the proof of Theorem 2 we obtain a complete bigraph $K_{r,2s}$.

Now we can give bounds for $\beta(B \times K_2)$.

**Theorem 4.** *Let $B$ be a connected finite bigraph with at least three vertices. Then*

$$\beta(B) + 2 \leqq \beta(B \times K_2) \leqq 2\,\beta(B)\,,$$

*and these bounds cannot be improved.*

Proof. Let the notation be the same as in the proof of Theorem 2. If we identify all vertices of $U''$ and all vertices of $V''$, we obtain a graph isomorphic to the graph obtained from $B$ by adding two vertices and an edge which joins them, joining one of them with all vertices of $U$ and the other with all vertices of $V$. By Theorem 1 the bichromaticity of this graph is $\beta(B) + 2$ and thus the lower bound is obtained. The case $B \cong K_{2,s}$ investigated in the proof of Theorem 2 shows that this bound cannot be improved. Let $K_{p,q}$ be a complete bigraph with $p + q = \beta(B \times K_2)$ onto which $B \times K_2$ can be mapped by a bicomplete homomorphism $\psi$. The vertex set of this graph is the union of the vertex sets of $\psi(B')$ and $\psi(B'')$. Each of these sets has the cardinality at most $\beta(B)$, therefore $\beta(B \times K_2) \leqq 2\,\beta(B)$. Consider the path $P_n$ with $n$ vertices, $n$ even. By Corollary 1a from [1] we have $\beta(P_n) = [\frac{1}{2}(n + 3)] = \frac{1}{2}n + 1$. Let the vertices of this path be $u_1, \ldots, u_n$ and the edges $u_iu_{i+1}$ for $i = 1, \ldots, n - 1$. Now $P_n \times K_2$ is the graph with the vertex set $\{u_1', \ldots, u_n', u_1'', \ldots, u_n''\}$ and with the edges $u_i'u_{i+1}', u_i''u_{i+1}''$ for $i = 1, \ldots, n - 1$ and $u_i'u_i''$ for $i = 1, \ldots, n$. This is a bigraph on the sets $U = \{u_i' \mid i \equiv 1 \pmod 2\} \cup \{u_i'' \mid i \equiv 0 \pmod 2\}$, $V = \{u_i' \mid i \equiv 0 \pmod 2\} \cup \{u_i'' \mid i \equiv 1 \pmod 2\}$. There exists a bicomplete homomorphism $\psi$ onto a complete bigraph $K_{2,n}$ such that $\psi(x) = \psi(y)$ if and only if either $x = y$ or $\{x, y\} \subset \{u_i' \mid i \equiv 1 \pmod 2\}$ or $\{x, y\} \subset \{u_i'' \mid i \equiv 0 \pmod 2\}$. Therefore $\beta(P_n \times K_2) = n + 2 = 2\,\beta(P_n)$ and the upper bound cannot be improved.

**Theorem 5.** *Let $B$ be a connected finite bigraph on the sets $U$, $V$. Let there exist an automorphism of $B$ which maps $U$ onto $V$ and $V$ onto $U$. Then*

$$\tfrac{3}{2}\mu(B) \leqq \beta(B \times K_2) \leqq 2\beta(B)\,.$$

This is an immediate consequence of Theorems 3 and 4.

As we see, the number $\beta(B \times K_2)$ is not uniquely determined by the number $\beta(B)$. Therefore our results cannot yield a formula for the bichromaticity of the graph of the cube of dimension $n$ as a function of $n$. We shall give only partial results.

**Theorem 6.** *Let $Q_n$ denote the graph of the cube of dimension n. Then*

$$\beta(Q_1) = 2\,, \quad \beta(Q_2) = 4\,, \quad \beta(Q_3) = 6\,, \quad \beta(Q_n) \geqq 2^{n-1} + 4 \textit{ for } n \geqq 4\,.$$

Proof. The computation of $\beta(Q_1)$, $\beta(Q_2)$ and $\beta(Q_3)$ is left to the reader. Now consider $Q_4$ as a bigraph on the sets $U$, $V$. To each vertex $x$ of $Q_4$ there exists a unique vertex $\bar{x}$ corresponding to the vertex of the cube which is opposite to the vertex of the cube corresponding to $x$. If $x \in U$, then also $\bar{x} \in U$. By identifying each pair $\{x, \bar{x}\}$ for $x \in U$ we obtain the complete bigraph $K_{4,8}$. Using Theorem 3 we prove by induction that $Q_n$ for each $n \geqq 4$ can be mapped by a bicomplete homomorphism onto $K_{4,s}$, where $s = 2^{n-1}$. Therefore $\beta(Q_n) \geqq 2^{n-1} + 4$ for each $n \geqq 4$.

Now we shall add some more results on the bichromaticity of a graph.

**Theorem 7.** *Let $B$ be a finite bigraph obtained from the complete bigraph $K_{n,n}$, where $n \geqq 3$, by deleting edges of a complete matching of $K_{n,n}$. Then*

$$\beta(B) = [\tfrac{3}{2}n]\,.$$

Proof. Let $\psi$ be a bicomplete homomorphism of $B$ onto $K_{r,s}$ for some $r$ and $s$. Consider a bijection $\gamma : U \to V$ such that for each $x \in U$ the vertex $\gamma(x)$ is the unique vertex of $V$ non-adjacent to $x$. If $x \in U$, then there exists $y \in U$ such that either $\psi(y) = \psi(x)$ or $\psi\,\gamma(y) = \psi\,\gamma(x)$; otherwise $\psi(x)$ and $\psi\,\gamma(x)$ would not be adjacent. We can define a mapping $\delta : U \to U$ so that if $x \in U$, then we put $\delta(x)$ equal to a vertex $y$ with the property that $\psi(y) = \psi(x)$, or $\psi\,\gamma(y) = \psi\,\gamma(x)$. This mapping defines a graph $H$ on the vertex set $U$ such that two vertices $x$ and $y$ are adjacent if and only if $y = \delta(x)$ or $x = \delta(y)$. If $x$ and $y$ are adjacent in $H$, then the four vertices $x, y, \gamma(x), \gamma(y)$ are mapped by $\psi$ onto at most three vertices. Therefore the difference between the number of vertices of $B$ and $r + s$ is equal at least to the number of edges of $H$. The graph $H$ must be a graph without isolated vertices. Hence the minimal number of edges of $H$ is $\frac{1}{2}n$ for $n$ even and $\frac{1}{2}(n + 1)$ for $n$ odd. We have proved that $\beta(B) \leqq 2n - \frac{1}{2}n = \frac{3}{2}n$ for $n$ even and $\beta(G) \leqq 2n - \frac{1}{2}(n + 1) = = \frac{3}{2}n - \frac{1}{2} = [\frac{3}{2}n]$ for $n$ odd. The equality can be proved by showing the corresponding bicomplete homomorphism $\psi$. We choose a partition $\mathscr{P}$ of $U$ such that if $n$ is even, then each class of $\mathscr{P}$ consists of two elements, and if $n$ is odd, then one class of $\mathscr{P}$ consists of three elements and each other class consists of two elements. The number of classes of $\mathscr{P}$ is $[\frac{1}{2}n]$. Now we can define a bicomplete homomorphism $\psi$ of $B$ onto $K_{r,s}$, where $r = [\frac{1}{2}n]$, $s = n$ so that $\psi(x) = \psi(y)$ if and only if either $x = y$ or $x$ and $y$ belong to the same class of $\mathscr{P}$.

**Theorem 8.** *Let $B$ be a finite bigraph obtained from the complete bigraph $K_{n,n}$ for even $n \geqq 4$ by deleting all edges of a Hamiltonian circuit of $K_{n,n}$. Then*

$$\beta(B) = \tfrac{3}{2}n\,.$$

Proof. As the graph $B$ from this theorem is a spanning subgraph of the graph $B$ from Theorem 7, its bichromaticity cannot be greater than $\frac{3}{2}n$. Therefore it suffices to describe a bicomplete homomorphism $\psi$ of $B$ onto $K_{r,s}$, where $r = \frac{1}{2}n$, $s = n$. Let $C$ be the mentioned Hamiltonian circuit of $K_{n,n}$. To each vertex $u \in U$ there exists exactly one vertex $\bar{u} \in U'$ which is opposite to $u$ in $C$. The neighbourhoods of $u$ and $\bar{u}$ in $C$ are disjoint, therefore we may define the bicomplete homomorphism $\psi$ of $B$ onto $K_{r,s}$ so that $\psi(x) = \psi(y)$ if and only if either $x = y$ or $x \in U$, $y \in U$, $\bar{x} = y$.

In the end we shall give a result on infinite bigraphs. For infinite connected bigraphs the bichromaticity can be defined analogously as for finite ones.

**Theorem 9.** *The bichromaticity of an infinite connected bigraph is equal to the cardinality of its vertex set.*

Proof. Let $B$ be an infinite connected bigraph on the sets $U$, $V$. The bichromaticity of $B$ evidently cannot be greater than the cardinality of its vertex set. Without loss of generality let $|U| \geqq |V|$. As $U \cup V$ is infinite, also $U$ is infinite and $|U| = |U \cup V|$. If $\psi$ is a bicomplete homomorphism of $B$ such that $\psi(x) = \psi(y)$ if and only if either $x = y$ or $\{x, y\} \in V$, then $\psi$ maps $B$ onto a star with the vertex set of the same cardinality as the vertex set of $B$ and the assertion is proved.

This theorem shows that the considerations on the bichromaticity of an infinite bigraph are trivial. Nonetheless, it might be interesting to study the pairs $\{r, s\}$ with the property that $B$ can be mapped by a bicomplete homomorphism onto $K_{r,s}$.

*Reference*

[1] *Harary, F., Hsu, D., Miller, Z.:* The bichromaticity of a tree. In: Theory and Applications of Graphs, ed. by Y. Alavi and D. R. Lick, Springer Verlag Berlin—Heidelberg—New York 1978, pp. 236—246.

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