Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 1, 48–52

Persistent URL: http://dml.cz/dmlcz/101721

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

A CHARACTERIZATION OF 0-MINIMAL (m, n)-IDEALS

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(Received February 27, 1979)

In [2], Kapp defined an equivalence relation B on a semigroup and used it to characterize 0-minimal bi-ideals. (see p. 84 in [1] for a definition of bi-ideals). In this paper we define equivalence relations B_m^n for m and n non-negative integers and use these relations to characterize 0-minimal (m, n)-ideals. For $m, n \geqq 1$ we have $B_m^n \subseteq B$.

Kapp also showed that if R $[L]$ is a 0-minimal right- [left-] ideal of a semigroup S, then either $RL = \{0\}$ or RL is a 0-minimal bi-ideal. We present here four generalizations of this result in section 2.

S will always denote a semigroup with zero element 0 unless stated otherwise.

1. CHARACTERIZATION OF 0-MINIMAL (m, n)-IDEALS

Definition (1.0). [See Def. 1.1 in [4]] A subsemigroup A of S is called an (m, n)-*ideal of* S if $A^m S A^n \subseteq A$, where m and n are non-negative integers.

Definition (1.1) For $a, b \in S$ (for any semigroup S) we write $a B_m^n b$ if and only if either 1) $a = b$ or 2) There exist $u, v \in S$ such that $a^m\, u\, a^n = b$ and $b^m\, v\, b^n = a$, where m and n are non-negative integers.

The following two propositions can be readily verified:

Proposition (1.2) *The relation B_m^n is an equivalence relation. Moreover, $B_m^n \subseteq B$ if $m, n \geqq 1$, where B is the equivalence relation defined by Kapp in* [2].

Proposition (1.3) *If A is an (m, n)-ideal of S, then $A = \bigcup_{a \in A} B_m^n(a)$, i.e., any (m, n)-ideal is the union of its B_m^n-classes. $B_m^n(a)$ is the B_m^n class containing a.*

Definition (1.4) A non-zero (m, n)-ideal A of S is said to be 0-*minimal* if there is no (m, n)-ideal A' of S such that $\{0\} \neq A' \subsetneqq A$.

Corollary (1.5). (to proposition (1.3)). *Let B be an (m, n)-ideal of S. If B is a single non-zero B_m^n-class union $\{0\}$, then B is a 0-minimal (m, n)-ideal of S.*

Lemma (1.6). *Let $a, b \in S$. Then $aB_m^n b$ if and only if $B_m^n(a) = B_m^n(b)$. That is to say, $aB_m^n b$ if and only if a and b generate the same principle (m, n)-ideal.*

Proof. Suppose $aB_m^n b$. If $a = b$, there is nothing to prove, so we can assume that $a \neq b$. Then there exist elements $u, v \in S$ such that $a = b^m u b^n$ and $b = a^m v a^n$. Note that $a^k = (b^m u b^n)^k \in b^m S b^n \subseteq B_m^n(b)$ for each k, $1 \leqq k \leqq m + n$. Moreover, $a^m S a^n = (b^m u b^n)^m S (b^m u b^n)^n \subseteq b^m S b^n \subseteq B_m^n(b)$. Thus, $B_m^n(a) \subseteq B_m^n(b)$. By a dual argument we can show that $B_m^n(b) \subseteq B_m^n(a)$.

Conversely, suppose $B_m^n(a) = B_m^n(b)$. Again, we can assume $a \neq b$. There are four cases to consider.

Case 1. $a = b^k$ for some k, $2 \leqq k \leqq m + n$, and $b \in a^m S a^n$. Then, there exists $u \in S$ such that $b = a^m u a^n = b^{mk} u b^{nk}$ and $a = b^k = (b^{mk} u b^{nk})^k \in b^m S b^n$. Therefore, we have $aB_m^n b$.

Case 2. $a = b^k$, and $b = a^l$ for some k and l between 2 and $m + n$, (since $a \neq b$). This implies that $a = b^k = a^{lk} = a^{lk^2} = a^{l^2 b^2} = \dots b^{l^r k^{r+1}} = \dots$. Thus, we can chose an r so that $l^r k^{r+1} > m + n + 1$, which implies that $a \in b^m S b^n$. Similarly, we can show that $b \in a^m S a^n$ and thus, $aB_m^n b$.

Case 3. $a \in b^m S b^n$ and $b = a^l$ for some l, $2 \leqq l \leqq m + n$.
This is simply the dual of case 1.

Case 4. $a \in b^m S b^n$ and $b \in a^m S a^n$.

Obviously, $aB_m^n b$.

Therefore, in all cases, we have that if $B_m^n(a) = B_m^n(b)$, then $aB_m^n b$.

Note that lemma 1.6 could be used to define the equivalence relations B_m^n in a way that generalized Green's relations L, R and J.

Theorem (1.7). *An (m, n)-ideal A of S is 0-minimal if and only if it is one non-zero B_m^n-class union $\{0\}$.*

Proof. By corollary (1.5), if A is one non-zero B_m^n-class union $\{0\}$, then A is a 0-minimal (m, n)-ideal.

Conversely, assume that A is a 0-minimal (m, n)-ideal. Let $a, b \in A \setminus \{0\}$. Again we can assume $a \neq b$. Let $B = B_m^n(b)$ and $C = B_m^n(a)$. Since $B \neq 0$, $C \neq 0$ and $B \subseteq A$, $C \subseteq A$, we have $B = A = C$ because A is 0-minimal. But, then by lemma 1.6, we have $aB_m^n b$. Thus, A is just one non-zero B_m^n-class union $\{0\}$.

Proposition (1.8). *Let I be a 0-minimal (m, n)-ideal. If $I^2 \neq 0$, then I is also a 0-minimal bi-ideal, (with $m, n \geqq 1$).*

Proof. Case 1. There exists a bi-ideal J of S such that $0 \neq J \subseteq I$. Then, since J is also an (m, n)-ideal, we have $J = I$ since I is 0-min (m, n)-ideal. But then I is a bi-ideal and in fact, a 0-minimal bi-ideal.

Case 2. There do not exist any bi-ideals J of S such that $0 \neq J \subseteq I$. Since $0 \neq I^2 \subseteq I$ and I is a 0-minimal (m, n)-ideal, we have $I^2 = I$. Thus, $I S I = I^m S I^n \subseteq I \Rightarrow I$ is a bi-ideal, and by the hypothesis of case 2, I must be a 0-minimal bi-ideal.

Corollary (1.9) (to proposition (1.8)) *A 0-minimal (m, n)-ideal A of S is either null or a group union $\{0\}$, $(m, n \geqq 1)$.*

Proof. If $A^2 = 0$, we are done. If $A^2 \neq 0$, then proposition (1.8) implies A is a 0-minimal bi-ideal and theorem 1.8 in [2] yields the desired result.

The following example will show that despite the similarity between our corollary (1.9) and theorem 1.8 in [2], the class of 0-minimal bi-ideals and the class of 0-minimal (m, n)-ideals are distinct.

Example (1.10). Let N be the non-negative integers, and $T = N/(6)$ be the set N mod 6. We will denote the elements of T by the symbols 0, 1, 2, 3, 4, 5.

Let

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ;\ a, b, c, d \in T \right\}.$$

Then S is a semigroup under multiplication with zero element

$$\bar{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Let

$$J = \left\{ \bar{0}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} \right\}.$$

Then $J^2 = \{\bar{0}\} \subseteq J$ and

$$J S J = \left\{ \bar{0}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} \right\} \subseteq J$$

imply that J is a bi-ideal. Moreover, since

$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} S \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} S \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

we have that J is a single non-zero B-class union $\{\bar{0}\}$, and so by corollary 1.6 in [2], J is a $\bar{0}$-minimal bi-ideal of S.

However, we can choose $\{\bar{0}\} \neq K = \left\{\bar{0}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\right\} \subsetneqq J$ and have that $K^2 = \{\bar{0}\}$ and hence $K^2SK = \{\bar{0}\} \subseteq K$. Therefore, K is a proper, non-zero (2.1)-ideal contained in J, and so J is not a $\bar{0}$-minimal (2,1)-ideal.

Moreover, K is a 0-minimal (2.1)-ideal which is not a bi-ideal, and hence not a 0-minimal bi-ideal. Thus, the class of 0-minimal bi-ideals and the class of 0-minimal (m, n)-ideals are distinct.

2. FACTORING A 0-MINIMAL (m, n)-IDEAL

In [2], proposition (1.9), it is shown that if R $[L]$ is a 0-minimal right-[left-] ideal of S, then either $RL = \{0\}$ or RL is a 0-minimal bi-ideal of S. The following four propositions represent an attempt to obtain a generalization of this result.

Proposition (2.1). *If S has the property that it contains no non-zero nilpotent (m, n)-ideals, and if R $[L]$ is a 0-minimal right - [left-] ideal of S, then either $RL = \{0\}$ or RL is a 0-minimal (m, n)-ideal of S.*

Proof. If $RL \neq \{0\}$, then by proposition (1.9) in [2] we have RL is a 0-minimal bi-ideal, and hence it is also an (m, n)-ideal. It remains to show that RL is a 0-minimal (m, n)-ideal.

Let $\{0\} \neq A \subseteq RL$ be an (m, n)-ideal of S. Note that since $RL \subseteq R \cap L$ we have $A \subseteq R \cap L$ and hence $A \subseteq R$ and $A \subseteq L$. By hypothesis, $A^m \neq \{0\}$ and $A^n \neq \{0\}$. Thus $\{0\} \neq A^mS^1 \subseteq R \Rightarrow A^mS^1 = R$ since R is 0-minimal. Also, $\{0\} \neq S^1A^n \subseteq$ $\subseteq L \Rightarrow S^1A^n = L$ since L is 0-minimal. Therefore, $A \subseteq RL = (A^mS^1)(S^1A^n) \subseteq$ $\subseteq A^mS^1A^n = A^{m+n} \cup A^mSA^n \subseteq A$ since A is an (m, n)-ideal. Thus $A = RL$, which means RL is a 0-minimal (m, n)-ideal.

Proposition (2.2). *Let R $[L]$ be a 0-minimal right- [left-] ideal of S. If R^mL^n is a subset of the center of S, then either $R^mL^n = \{0\}$ or R^mL^n is a 0-minimal (m, n)-ideal.*

Proof. If $R^mL^n \neq \{0\}$, then $R^m \neq \{0\}$ and $L^n \neq \{0\}$, and hence $\{0\} \neq R^m \subseteq R \Rightarrow$ $\Rightarrow R^m = R$ and $\{0\} \neq L^n \subseteq L \Rightarrow L^n = L$ since R $[L]$ is a 0-minimal right- [left-] ideal of S. Thus, $R^mL^n = RL$ is a 0-minimal bi-ideal by proposition (1.9) in [2], and hence is also an (m, n)-ideal. Now we show that R^mL^n is 0-minimal. Let $\{0\} \neq A \subseteq$ $\subseteq R^mL^n = RL \subseteq R \cap L$ be an (m, n)-ideal of S. Then $A \subseteq R$ and $A \subseteq L \Rightarrow \{0\} \neq$ $\neq AS^1 \subseteq RS^1 \subseteq R$ and $\{0\} \neq S^1A \subseteq S^1L \subseteq L$ and thus $AS^1 = R$ and $S^1A = L$ since R $[L]$ is a 0-minimal right- [left-] ideal. Therefore, $A \subseteq R^mL^n = (AS^1)^m \cdot$ $\cdot (S^1A)^n = A^m(S^1)^{m+n}A^n \subseteq A^mS^1A^n = A^{m+n} \cup A^mSA^n \subseteq A$ since A is in the center of S and is an (m, n)-ideal of S. This means that $A = R^mL^n$ and so R^mL^n is a 0-minimal (m, n)-ideal.

We conclude this paper with two propositions that use theorem 2 in [3] which says that S is (m, n)-regular if and only if $I = I^m S I^n$ for every (m, n)-ideal I of S.

Proposition (2.3). *If S is (m, n)-regular, and if A $[B]$ is a 0-minimal $(m, 0)$-$[(0, n)$-$]$ ideal such that $AB \subseteq A \cap B$, then either $AB = \{0\}$ or AB is a 0-minimal (m, n)-ideal.*

Proof. Let $C = AB$. If $C \neq \{0\}$, then $C^2 = (AB)(AB) \subseteq (AB) B \subseteq AB = C$. Moreover, $C^m S C^n = (AB)^m S(AB)^n \subseteq (A^m S) B^n \subseteq AB^n \subseteq AB = C$. Thus, C is a subsemigroup such that $C^m S C^n \subseteq C$, i.e., C is an (m, n)-ideal.

Let $\{0\} \neq D \subseteq C$ be a nonzero (m, n)-ideal. Then since S is (m, n)-regular we have $\{0\} \neq D = D^m S D^n$ and hence $D^m S \neq \{0\}$ and $SD^n \neq \{0\}$. Further, $D \subseteq C = AB \subseteq A \cap B \Rightarrow D \subseteq A$ and $D \subseteq B$, therefore, $\{0\} \neq D^m S \subseteq A^m S \subseteq A$ since A is an $(m, 0)$-ideal, and $D^m S = A$ since A is 0-minimal. Likewise, $\{0\} \neq SD^n \subseteq B \Rightarrow SD^n = B$. So we have

$$D \subseteq AB = (D^m S)(SD^n) \subseteq D^m S D^n = D\,.$$

This means $D = AB$ and hence AB is 0-minimal.

Proposition (2.4). *If S is (m, n)-regular, and if A $[B]$ is a 0-minimal $(m, 0)$-$[(0, n)$-$]$ ideal, then either $A \cap B = \{0\}$ or $A \cap B$ is a 0-minimal (m, n)-ideal.*

Proof. Once we establish that $A \cap B$ is an (m, n)-ideal, the rest of the proof is the same as in (2.3) above.

Let $C = A \cap B$, then $C^2 \subseteq A^2 \subseteq A$ and $C^2 \subseteq B^2 \subseteq B$. Hence, $C^2 \subseteq A \cap B = C$. So C is a subsemigroup.

$C^m S C^n \subseteq (A^m S) B^n \subseteq AB^n \subseteq SB^n \subseteq B$. But, we also have $C^m S C^n \subseteq A^m (SB^n) \subseteq A^m B \subseteq A^m S \subseteq A$. Thus, $C^m S C^n \subseteq A \cap B = C$ and so C is a nonzero (m, n)-ideal.

References

[1] *Clifford, A. H.* and *Preston, G. B.:* The Algebraic theory of semigroups, Math. Surveys of the American Math. Soc. 7 Vol. 1 (Providence, R.I., 1961).

[2] *Kapp, K. M.:* On Bi-ideals and Quasi-ideals in Semigroups, Publ. Math. Debrecen, *16* (1969), 179—185.

[3] *Krgovic, D. N.:* On (m, n)-regular Semigroups, Publ. Inst. Math. Belgrade *18 (32)* (1975), 107—110.

[4] *Lajos, Sandor:* Generalized Ideals in Semigroups, Acta Sci. Math., *22* (1961), 217—222.

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