Czechoslovak Mathematical Journal

Josef Janyška

On linear functions on the sphere S^2

Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 1, 75–82

Persistent URL: http://dml.cz/dmlcz/101724

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

ON LINEAR FUNCTIONS ON THE SPHERE S^2

JOSEF JANYŠKA, Brno

(Received April 9, 1979)

1. Let S^2 be a unit sphere in E^3. Let D be a domain in S^2. A function $\mathbf{f} : D \to R$ is called *linear* if

$$f(M) = \langle \mathbf{m}, \mathbf{a} \rangle + k , \tag{1}$$

where $\mathbf{a}$ is a constant vector, $\mathbf{m}$ is the position vector of the point $M \in S^2$ with respect to the centre of S^2, $k \in R$, and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in E^3. The linear function $\mathbf{f}$ is called *homogeneous* or *non-homogeneous* if $k = 0$ or $k \neq 0$, respectively.

In [1], A. Švec found certain conditions for a function $\mathbf{f}$ to be linear and homogeneous. These conditions are expressed in terms of partial differential equations on D or on the boundary ∂D of D.

The aim of this paper is to extend the results obtained by A. Švec to the wider class of non-homogeneous linear functions on the domains in S^2.

2. Let us introduce some notations (see [1]). Consider the unit sphere $S^2 \subset E^3$. With each point M of S^2, let us associate a tangent orthonormal frame $\{\mathbf{m}, v_1, v_2, v_3\}$ such that $\mathbf{m}$ is the position vector of the point $M \in S^2$, v_1, v_2 are tangent vectors to S^2 at M, and v_3 is a normal vector to S^2 at M. Then we have

$$\mathrm{d}\mathbf{m} = \omega^1 v_1 + \omega^2 v_2 , \quad \mathrm{d}v_1 = \omega_1^2 v_2 + \omega^1 v_3 ,$$
$$\mathrm{d}v_2 = -\omega_1^2 v_1 + \omega^2 v_3 , \quad \mathrm{d}v_3 = -\omega^1 v_1 - \omega^2 v_2 .$$

Let $\mathbf{f} : S^2 \to R$ be a function. Recall that the covariant derivatives $\mathbf{f}_i, \mathbf{f}_{ij}, P, \ldots, S, T_1, \ldots, T_5$ $(i, j = 1, 2)$ of $\mathbf{f}$ with respect to a field of tangent orthonormal frames $\{\mathbf{m}, v_k\}$ $(k = 1, 2, 3)$ are defined by the following formulas:

$$\mathrm{d}\mathbf{f} = \mathbf{f}_1\omega^1 + \mathbf{f}_2\omega^2 ; \tag{2}$$

$$\mathrm{d}\mathbf{f}_1 - \mathbf{f}_2\omega_1^2 = \mathbf{f}_{11}\omega^1 + \mathbf{f}_{12}\omega^2 , \tag{3}$$
$$\mathrm{d}\mathbf{f}_2 + \mathbf{f}_1\omega_1^2 = \mathbf{f}_{12}\omega^1 + \mathbf{f}_{22}\omega^2 ;$$

(4) $$df_{11} - 2f_{12}\omega_1^2 = P\omega^1 + Q\omega^2 ,$$
$$df_{12} + (f_{11} - f_{22})\,\omega_1^2 = (Q + f_2)\,\omega^1 + (R + f_1)\,\omega^2 ,$$
$$df_{22} + 2f_{12}\omega_1^2 = R\omega^1 + S\omega^2 ;$$

(5) $$dP - (3Q + 2f_2)\,\omega_1^2 = T_1\omega^1 + T_2\omega^2 ,$$
$$dQ + (P - 2R - 2f_1)\,\omega_1^2 = (T_2 + 2f_{12})\omega^1 + (T_3 + 2f_{11})\,\omega^2 ,$$
$$dR + (2Q - S + 2f_2)\,\omega_1^2 = (T_3 + 2f_{22})\,\omega^1 + (T_4 + 2f_{12})\,\omega^2 ,$$
$$dS + (3R + 2f_1)\,\omega_1^2 = T_4\omega^1 + T_5\omega^2 .$$

By means of these covariant derivatives, one can introduce the following differential operators $\mathscr{L}$ and $\mathscr{M}$, which play an important role in investigations of the linearity conditions (1):

$$\mathscr{L}f = f_{11} + f_{22} + 2f ,$$
$$\mathscr{M}f = f_{11}f_{22} - f_{12}^2 + f(f_{11} + f_{22} + f) .$$

3. Let $D \subset S^2$ be a domain, ∂D the boundary of D, $\bar{D} = D \cup \partial D$, and let $f : \bar{D} \to R$ be a function. In all proofs, we shall use the following

Lemma. *If* $L = (\mathscr{L}f)^2 - 4\mathscr{M}f = 0$ *on* $\bar{D}$, *then* f *is linear on* $\bar{D}$.

Proof. Supposition $L = (\mathscr{L}f)^2 - 4\mathscr{M}f = (f_{11} - f_{22})^2 + 4f_{12}^2 = 0$ yields $f_{11} - f_{22} = f_{12} = 0$, and from (4) we get $P = R$, $Q = S$, $Q = -f_2$, $R = -f_1$. Then $df_{11} = df_{22} = P\omega^1 + Q\omega^2 = -f_1\omega^1 - f_2\omega^2 = -df$. This implies that $f_{11} = f_{22} = -f + c$, where c is an integral constant. Now, let us consider the vector field

(6) $$a = -f_1v_1 - f_2v_2 + (f - c)\,v_3$$

on $\bar{D}$. Then $da = 0$ and hence a is a constant vector. From (6) we get $f - c = \langle v_3, a\rangle$. QED.

4. Let $\mathscr{L}$, $\mathscr{M}$, L be given as in Sections 2 and 3. In the proofs of the following Theorems 1–8, we shall use the maximum principle in the form described in [2].

Theorem 1. *Let* $D \subset S^2$ *be a domain,* ∂D *its boundary and* $\bar{D} = D \cup \partial D$. *Let* $f : \bar{D} \to R$ *be a function. If*

1. $L = 0$ *on* ∂D,
2. $(f_{11} - f_{22})\,[(\mathscr{L}f)_{11} - (\mathscr{L}f)_{22}] + 4f_{12}(\mathscr{L}f)_{12} \geqq 0$ *on* D,

then f *is linear on* $\bar{D}$.

Proof. Consider the covariant derivatives of the functions $\mathscr{L}f$ and L. The formulas (2)–(5) immediately lead to the expressions

(7) $(\mathscr{L}f)_1 = P + R + 2f_1 , \quad (\mathscr{L}f)_2 = Q + S + 2f_2 ;$

$$(8) \qquad (\mathscr{L}\mathbf{f})_{11} = 2(\mathbf{f}_{11} + \mathbf{f}_{22}) + T_1 + T_3\,,$$
$$(\mathscr{L}\mathbf{f})_{12} = 4\mathbf{f}_{12} + T_2 + T_4\,,$$
$$(\mathscr{L}\mathbf{f})_{22} = 2(\mathbf{f}_{11} + \mathbf{f}_{22}) + T_3 + T_5\,;$$
$$(9) \qquad L_1 = 2(\mathbf{f}_{11} - \mathbf{f}_{22})(P - R) + 8\mathbf{f}_{12}(Q + \mathbf{f}_2)\,,$$
$$L_2 = 2(\mathbf{f}_{11} - \mathbf{f}_{22})(Q - S) + 8\mathbf{f}_{12}(R + \mathbf{f}_1)\,;$$
$$(10) \qquad L_{11} = 2(P - R)^2 + 8(Q + \mathbf{f}_2)^2 - 4\mathbf{f}_{22}(\mathbf{f}_{11} - \mathbf{f}_{22}) + 24\mathbf{f}_{12}^2 +$$
$$+ 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,T_1 + 8\mathbf{f}_{12}T_2 - 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,T_3\,,$$
$$L_{12} = 2(P - R)(Q - S) + 8(Q + \mathbf{f}_2)(R + \mathbf{f}_1) + 12\mathbf{f}_{12}(\mathbf{f}_{11} + \mathbf{f}_{22}) +$$
$$+ 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,T_2 + 8\mathbf{f}_{12}T_3 - 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,T_4\,,$$
$$L_{22} = 2(Q - S)^2 + 8(R + \mathbf{f}_1)^2 + 4\mathbf{f}_{11}(\mathbf{f}_{11} - \mathbf{f}_{22}) + 24\mathbf{f}_{12}^2 +$$
$$+ 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,T_3 + 8\mathbf{f}_{12}T_4 - 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,T_5\,.$$

Eliminating $T_1, \ldots, T_5$ from (8) nad (10), one obtains

$$L_{11} + L_{22} - 4L = 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,[(\mathscr{L}\mathbf{f})_{11} - (\mathscr{L}\mathbf{f})_{22}] + 8\mathbf{f}_{12}(\mathscr{L}\mathbf{f})_{12} +$$
$$+ 8(Q + \mathbf{f}_2)^2 + 8(R + \mathbf{f}_1)^2 + 2(P - R)^2 + 2(Q - S)^2\,.$$

Now we can conclude from 2. that this expression satisfies the conditions of the maximum principle for the function L. Thus 1. implies $L = 0$ on $\bar{D}$ and the theorem follows from Lemma. QED.

Theorem 2. *Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f} : \bar{D} \to R$ be a function. If*

1. $L = 0$ *on* ∂D,
2. $(\mathbf{f}_{11} - \mathbf{f}_{22})\,[(\mathscr{M}\mathbf{f})_{11} - (\mathscr{M}\mathbf{f})_{22}] + 4\mathbf{f}_{12}(\mathscr{M}\mathbf{f})_{12} \geqq 0$ *on* D,
3. $\mathscr{M}\mathbf{f} > 0$, $\mathscr{L}\mathbf{f} \geqq 0$ *on* D,

then $\mathbf{f}$ *is linear on* $\bar{D}$.

Proof. Consider the covariant derivatives of the function $\mathscr{M}\mathbf{f}$. We directly obtain from (2)–(5) the identities

$$(11) \qquad (\mathscr{M}\mathbf{f})_1 = (\mathbf{f}_{22} + \mathbf{f})(P + \mathbf{f}_1) + (\mathbf{f}_{11} + \mathbf{f})(R + \mathbf{f}_1) - 2\mathbf{f}_{12}(Q + \mathbf{f}_2)\,,$$
$$(\mathscr{M}\mathbf{f})_2 = (\mathbf{f}_{22} + \mathbf{f})(Q + \mathbf{f}_2) + (\mathbf{f}_{11} + \mathbf{f})(S + \mathbf{f}_2) - 2\mathbf{f}_{12}(R + \mathbf{f}_1)\,;$$

$$(12)\qquad \begin{aligned}(\mathcal{M}\mathbf{f})_{11} &= 2(P + \mathbf{f}_1)(R + \mathbf{f}_1) - 2(Q + \mathbf{f}_2)^2 + \mathcal{L}\mathbf{f}\cdot\mathbf{f}_{11} - 6\mathbf{f}_{12}^2 + \\ &\quad + 2\mathbf{f}_{22}(\mathbf{f}_{11} + \mathbf{f}) + (\mathbf{f}_{22} + \mathbf{f})\,T_1 - 2\mathbf{f}_{12}T_2 + (\mathbf{f}_{11} + \mathbf{f})\,T_3\,, \\ (\mathcal{M}\mathbf{f})_{12} &= -(R + \mathbf{f}_1)(Q + \mathbf{f}_2) + (S + \mathbf{f}_2)(P + \mathbf{f}_1) + \mathcal{L}\mathbf{f}\cdot\mathbf{f}_{12} + \\ &\quad + 2\mathbf{f}_{12}(\mathbf{f} - \mathbf{f}_{11} - \mathbf{f}_{22}) + (\mathbf{f}_{22} + \mathbf{f})\,T_2 - 2\mathbf{f}_{12}T_3 + \\ &\quad + (\mathbf{f}_{11} + \mathbf{f})\,T_4\,, \\ (\mathcal{M}\mathbf{f})_{22} &= 2(Q + \mathbf{f}_2)(S + \mathbf{f}_2) - 2(R + \mathbf{f}_1)^2 + \mathcal{L}\mathbf{f}\cdot\mathbf{f}_{22} - 6\mathbf{f}_{12}^2 + \\ &\quad + 2\mathbf{f}_{11}(\mathbf{f}_{22} + \mathbf{f}) + (\mathbf{f}_{22} + \mathbf{f})\,T_3 - 2\mathbf{f}_{12}T_4 + (\mathbf{f}_{11} + \mathbf{f})\,T_5\,.\end{aligned}$$

(10) and (12) yield

$$\begin{aligned}&(\mathbf{f}_{22} + \mathbf{f})\,L_{11} - 2\mathbf{f}_{12}L_{12} + (\mathbf{f}_{11} + \mathbf{f})\,L_{22} - 2\mathcal{L}\mathbf{f}L = \\ &= 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,[(\mathcal{M}\mathbf{f})_{11} - (\mathcal{M}\mathbf{f})_{22}] + 8\mathbf{f}_{12}(\mathcal{M}\mathbf{f})_{12} + \\ &+ 2(\mathbf{f}_{11} + \mathbf{f})(Q + S + 2\mathbf{f}_2)^2 - 4(P + R + 2\mathbf{f}_1)(Q + S + 2\mathbf{f}_2)\,\mathbf{f}_{12} + \\ &+ 2(\mathbf{f}_{22} + \mathbf{f})(P + R + 2\mathbf{f}_1)^2 + 4(\mathcal{L}\mathbf{f})\,[(R + \mathbf{f}_1)(R - P) + (Q + \mathbf{f}_2)(Q - S)]\,.\end{aligned}$$

From (9) we get

$$\begin{aligned}&(R + \mathbf{f}_1)\,L_1 - (Q + \mathbf{f}_2)\,L_2 = \\ &= -2(\mathbf{f}_{11} - \mathbf{f}_{22})\,[(R + \mathbf{f}_1)(R - P) + (Q + \mathbf{f}_2)(Q - S)]\end{aligned}$$

and hence $[(R + \mathbf{f}_1)(R - P) + (Q + \mathbf{f}_2)(Q - S)] = -\varrho L_1 + \sigma L_2$, where ϱ, σ satisfy $2(\mathbf{f}_{11} - \mathbf{f}_{22})\,\varrho = R + \mathbf{f}_1$, $2(\mathbf{f}_{11} - \mathbf{f}_{22})\,\sigma = Q + \mathbf{f}_2$. Thus we have

$$(13)\qquad \begin{aligned}&(\mathbf{f}_{22} + \mathbf{f})\,L_{11} - 2\mathbf{f}_{12}L_{12} + (\mathbf{f}_{11} + \mathbf{f})\,L_{22} + 4\mathcal{L}\mathbf{f}\varrho L_1 - 4\mathcal{L}\mathbf{f}\sigma L_2 - 2\mathcal{L}\mathbf{f}L = \\ &= 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,[(\mathcal{M}\mathbf{f})_{11} - (\mathcal{M}\mathbf{f})_{22}] + 8\mathbf{f}_{12}(\mathcal{M}\mathbf{f})_{12} + \\ &+ 2(\mathbf{f}_{11} + \mathbf{f})(Q + S + 2\mathbf{f}_2)^2 - 4\mathbf{f}_{12}(P + R + 2\mathbf{f}_1)(Q + S + 2\mathbf{f}_2) + \\ &+ 2(\mathbf{f}_{22} + \mathbf{f})(P + R + 2\mathbf{f}_1)^2\,.\end{aligned}$$

The quadratic form on the right hadn side is positive definite because of 3. This implies that the expression (13) satisfies the conditions of the maximum principle for the function L, and we must have $L = 0$ on $\bar{D}$. The theorem now follows from Lemma. QED.

Theorem 3. *Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f} : \bar{D} \to R$ be a function. If*

1. $L = 0$ *on* ∂D,
2. $\mathcal{L}\mathbf{f}[(\mathcal{L}\mathbf{f})_{11} + (\mathcal{L}\mathbf{f})_{22}] - 2[(\mathcal{M}\mathbf{f})_{11} + (\mathcal{M}\mathbf{f})_{22}] \geqq 0$ *on* D,
   then $\mathbf{f}$ *is linear on* $\bar{D}$.

Proof. From (8), (10) and (12) we get

$$L_{11} + L_{22} = 2\mathscr{L}f[(\mathscr{L}f)_{11} + (\mathscr{L}f)_{22}] - 4[(\mathscr{M}f)_{11} + (\mathscr{M}f)_{22}] + $$
$$+ 2(P + R + 2f_1)^2 + 2(Q + S + 2f_2)^2 .$$

This expression satisfies the conditions of the maximum principle for the function L, and we must have $L = 0$ on $\bar{D}$. The theorem now follows from Lemma. QED.

Theorem 4. *Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $f : \bar{D} \to R$ be a function. If*

1. $L = 0$ *on* ∂D,
2. $\mathscr{L}f[(f_{22} + f)(\mathscr{L}f)_{11} - 2f_{12}(\mathscr{L}f)_{12} + (f_{11} + f)(\mathscr{L}f)_{22}] - 2[(f_{22} + f)(\mathscr{M}f)_{11} - 2f_{12}(\mathscr{M}f)_{12} + (f_{11} + f)(\mathscr{M}f)_{22}] \geqq 0$ *on* D,
3. $\mathscr{M}f > 0$, $\mathscr{L}f \geqq 0$ *on* D,

then f is linear on $\bar{D}$.

Proof. From (8), (10) and (12) we get

$$(14) \qquad (f_{22} + f) L_{11} - 2f_{12}L_{12} + (f_{11} + f) L_{22} =$$
$$= 2\mathscr{L}f[(f_{22} + f)(\mathscr{L}f)_{11} - 2f_{12}(\mathscr{L}f)_{12} + (f_{11} + f)(\mathscr{L}f)_{22}] -$$
$$- 4[(f_{22} + f)(\mathscr{M}f)_{11} - 2f_{12}(\mathscr{M}f)_{12} + (f_{11} + f)(\mathscr{M}f)_{22}] +$$
$$+ 2(f_{22} + f)(P + R + 2f_1)^2 - 4f_{12}(P + R + 2f_1)(Q + S + 2f_2) +$$
$$+ 2(f_{11} + f)(Q + S + 2f_2)^2 .$$

The quadratic form on the right hand side is positive definite because of 3. This implies that the expression (14) satisfies the conditions of the maximum principle for the function L, and we must have $L = 0$ on $\bar{D}$. The theorem now follows from Lemma. QED.

5. Let us consider the orthonormal tangent vector fields V_1, V_2 on S^2. Let the tangent frames on S^2 be chosen in such a way that $v_i = V_i$ $(i = 1, 2)$. Put $v_i f = \mathrm{d}f(v_i)$.

Theorem 5. *Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $f : \bar{D} \to R$ be a function satisfying $f_{12} = 0$ on D. If*

1. $L = 0$ *on* ∂D,
2. *on D, there is a couple of orthonormal tangent vector fields V_1, V_2 such that*

$$(f_{11} - f_{22})(V_1V_1 - V_2V_2)\mathscr{L}f \geqq 0 ,$$

then f is linear on $\bar{D}$.

Proof. From $\mathrm{d}f_{12} = 0$ we get $(f_{11} - f_{22})\,\omega_1^2 = (Q + f_2)\,\omega^1 + (R + f_1)\,\omega^2$. Consequently, there are functions α, β such that

$$(15) \qquad \omega_1^2 = \alpha\omega^1 + \beta\omega^2 ,$$

and α, β satisfy $(f_{11} - f_{22})\alpha = Q + f_2$, $(f_{11} - f_{22})\beta = R + f_1$. We have $v_1\mathscr{L}f = P + R + 2f_1$, $v_2\mathscr{L}f = Q + S + 2f_2$. This together with (15) implies

$$v_1v_1\mathscr{L}f = 2(f_{11} + f_{22}) + (Q + S + 2f_2)\alpha + T_1 + T_3\,, \tag{16}$$

$$v_2v_2\mathscr{L}f = 2(f_{11} + f_{22}) - (P + R + 2f_1)\beta + T_3 + T_5\,.$$

Eliminating $T_1, \ldots, T_5$ from (10) and (16) we obtain

$$L_{11} + L_{22} - 4L = 2(f_{11} - f_{22})(v_1v_1 - v_2v_2)\mathscr{L}f + 2[(P + f_1) - \tfrac{3}{2}(R + f_1)]^2 + 2[(S + f_2) - \tfrac{3}{2}(Q + f_2)]^2 + \tfrac{7}{2}[(Q + f_2)^2 + (R + f_1)^2]\,.$$

Now we can conclude from 2. that this expression satisfies the conditions of the maximum principle for the function L, and $L = 0$ on $\bar{D}$. The theorem follows from Lemma. QED.

Theorem 6. *Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $f : \bar{D} \to R$ be a function satisfying $f_{12} = 0$ on D. If*

1. *$L = 0$ on ∂D,*
2. *on D. there is a couple of orthonormal tangent vector fields V_1, V_2 such that*

$$(f_{11} - f_{22})(V_1V_1 - V_2V_2)\mathscr{M}f \geqq 0\,,$$

3. *$\mathscr{M}f > 0$, $\mathscr{L}f \geqq 0$ on D,*
4. *$\dfrac{4}{11} \leqq \dfrac{(f_{22} + f)^2}{(f_{11} + f)^2} \leqq \dfrac{11}{4}$ on D,*

then f is linear on $\bar{D}$.

Proof. We have

$$v_1\mathscr{M}f = (f_{11} + f)(R + f_1) + (f_{22} + f)(P + f_1)\,,$$

$$v_2\mathscr{M}f = (f_{11} + f)(S + f_2) + (f_{22} + f)(Q + f_2)\,;$$

from (15), we obtain

$$v_1v_1\mathscr{M}f = 2(P + f_1)(R + f_1) - 2(Q + f_2)^2 + 2f_{22}(f_{11} + f) + \mathscr{L}f \cdot f_{11} + [(f_{11} + f)(S + f_2) + (f_{22} + f)(Q + f_2)]\alpha + (f_{22} + f)T_1 + (f_{11} + f)T_3\,, \tag{17}$$

$$v_2v_2\mathscr{M}f = 2(Q + f_2)(S + f_2) - 2(R + f_1)^2 + 2f_{11}(f_{22} + f) + \mathscr{L}f \cdot f_{22} - [(f_{11} + f)(R + f_1) + (f_{22} + f)(P + f_1)]\beta + (f_{22} + f)T_3 + (f_{11} + f)T_5\,.$$

Eliminating $T_1, \ldots, T_5$ from (10) and (17) we obtain

$$(\mathbf{f}_{22} + \mathbf{f})\, L_{11} + (\mathbf{f}_{11} + \mathbf{f})\, L_{22} - 2\mathscr{L}\mathbf{f}L = 2(\mathbf{f}_{11} - \mathbf{f}_{22})\,(v_1 v_1 - v_2 v_2)\,\mathscr{M}\mathbf{f} +$$

$$+ 2(\mathbf{f}_{11} + \mathbf{f}) \left\{ \left[(S + \mathbf{f}_2) - \left(\frac{1}{2} + \frac{\mathbf{f}_{22} + \mathbf{f}}{\mathbf{f}_{11} + \mathbf{f}} \right) (Q + \mathbf{f}_2 \right]^2 + \right.$$

$$\left. + \left(\frac{11}{4} - \frac{(\mathbf{f}_{22} + \mathbf{f})^2}{(\mathbf{f}_{11} + \mathbf{f})^2} \right) (Q + \mathbf{f}_2)^2 \right\} +$$

$$+ 2(\mathbf{f}_{22} + \mathbf{f}) \left\{ \left[(P + \mathbf{f}_1) - \left(\frac{1}{2} + \frac{\mathbf{f}_{11} + \mathbf{f}}{\mathbf{f}_{22} + \mathbf{f}} \right) (R + \mathbf{f}_1) \right]^2 + \right.$$

$$\left. + \left(\frac{11}{4} - \frac{(\mathbf{f}_{11} + \mathbf{f})^2}{(\mathbf{f}_{22} + \mathbf{f})^2} \right) (R + \mathbf{f}_1)^2 \right\} .$$

Now we can conclude from 2., 3. and 4. that this expression satisfies the conditions of the maximum principle for the function L, and $L = 0$ on $\bar{D}$. The theorem now follows from Lemma. QED.

Theorem 7. *Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f} : \bar{D} \to R$ be a function satisfying $\mathbf{f}_{12} = 0$ on D. If*

1. *$L = 0$ on ∂D,*
2. *on D, there is a couple of orthonormal tangent vector fields V_1, V_2 such that*

$$\mathscr{L}\mathbf{f}(V_1 V_1 + V_2 V_2)\, \mathscr{L}\mathbf{f} - 2(V_1 V_1 + V_2 V_2)\, \mathscr{M}\mathbf{f} \geqq 0 ,$$

then $\mathbf{f}$ is linear on $\bar{D}$.

Proof. From (10), (16) and (17) we obtain

$$L_{11} + L_{22} + \alpha L_2 - \beta L_1 = 2\mathscr{L}\mathbf{f}(v_1 v_1 + v_2 v_2)\, \mathscr{L}\mathbf{f} -$$

$$- 4(v_1 v_1 + v_2 v_2)\, \mathscr{M}\mathbf{f} + 2(P + R + 2\mathbf{f}_1)^2 + 2(Q + S + 2\mathbf{f}_2)^2 .$$

This expression satisfies the conditions of the maximum principle for the function L, and we must have $L = 0$ on $\bar{D}$. The theorem now follows from Lemma. QED.

Theorem 8. *Let $D \subset S^2$ be a domain, ∂D its boundary and $\bar{D} = D \cup \partial D$. Let $\mathbf{f} : \bar{D} \to R$ be a function satisfying $\mathbf{f}_{12} = 0$ on D. If*

1. *$L = 0$ on ∂D,*
2. *on D, there is a couple of orthonormal tangent vector fields V_1, V_2 such that*

$$\mathscr{L}\mathbf{f}[(\mathbf{f}_{22} + \mathbf{f})\, V_1 V_1 \mathscr{L}\mathbf{f} + (\mathbf{f}_{11} + \mathbf{f})\, V_2 V_2 \mathscr{L}\mathbf{f}] -$$

$$- 2[(\mathbf{f}_{22} + \mathbf{f})\, V_1 V_1 \mathscr{M}\mathbf{f} + (\mathbf{f}_{11} + \mathbf{f})\, V_2 V_2 \mathscr{M}\mathbf{f}] \geqq 0 ,$$

3. $\mathcal{M}\mathbf{f} > 0$, $\mathcal{L}\mathbf{f} \geqq 0$ *on* D,
then $\mathbf{f}$ *is linear on* $\bar{D}$.

Proof. From (10), (16) and (17) we obtain

$$(18) \quad L\ (\mathbf{f}_{22} + \mathbf{f}) L_{11} + (\mathbf{f}_{11} + \mathbf{f}) L_{22} - (\mathbf{f}_{11} + \mathbf{f}) \beta L_1 + (\mathbf{f}_{22} + \mathbf{f}) \alpha L_2 =$$

$$= 2\mathcal{L}\mathbf{f}[(\mathbf{f}_{22} + \mathbf{f}) v_1 v_1 \mathcal{L}\mathbf{f} + (\mathbf{f}_{11} + \mathbf{f}) v_2 v_2 \mathcal{L}\mathbf{f}] -$$

$$- 4[(\mathbf{f}_{22} + \mathbf{f}) v_1 v_1 \mathcal{M}\mathbf{f} + (\mathbf{f}_{11} + \mathbf{f}) v_2 v_2 \mathcal{M}\mathbf{f}] +$$

$$+ 2(\mathbf{f}_{22} + \mathbf{f})(P + R + 2\mathbf{f}_1)^2 + 2(\mathbf{f}_{11} + \mathbf{f})(Q + S + 2\mathbf{f}_2)^2 .$$

The quadratic form on the right hand side is positive definite because of 3. This implies that the expression (18) satisfies the conditions of the maximum principle for the function L, and we must have $L = 0$ on $\bar{D}$. The theorem now follows from Lemma. QED.

References

[1] *A. Švec:* A remark on the differential equations on the sphere, Časopis pro pěstování matematiky, roč. *101*, (1976), 278—282.

[2] *A. Švec:* Contributions to the global differential geometry of surfaces, Rozpravy ČSAV, vol. *87*, No. 1, Praha 1977.

Author's address: 611 37 Brno, Kotlářská 2, ČSSR (přírodovědecká fakulta UJEP).