Czechoslovak Mathematical Journal

František Neuman

On transformations of differential equations and systems with deviating argument

Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 1, 87–90

Persistent URL: `http://dml.cz/dmlcz/101726`

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

ON TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS AND SYSTEMS WITH DEVIATING ARGUMENT

FRANTIŠEK NEUMAN, Brno

(Received April 26, 1979)

I. Consider a differential equation or system $A_n(\xi(x))$ of the form

$$\mathscr{A}_i(x, y_1(x), y_1(\xi(x)), y_2(x), y_2(\xi(x)), \ldots, y_m(x), y_m(\xi(x)),$$
$$y_1'(x), y_1'(\xi(x)), \ldots, y_m^{(n)}(x), y_m^{(n)}(\xi(x))) = 0\,,$$

$i = 1, \ldots, m$, on an interval $I = (a, b) \subset \mathbb{R}$ with one (bounded or unbounded) deviating argument ξ. It is supposed $\xi \in C_n(a_1, b)$, $\xi'(x) > 0$ and $\xi(x) \neq x$ on (a_1, b). Moreover, $\xi(a_1, b) = (a, b)$, (i.e. $\xi(b) = b$), $\xi(a_1) = a$ for $\xi(x) < x$, and $a_1 = a$ for $\xi(x) > x$. We do not exclude $a = -\infty$, $a_1 = -\infty$, and $b = \infty$. With these restrictions, the system $A_n(\xi(x))$ includes both linear and nonlinear, retarded, advanced, and neutral differential systems as considered, e.g., in [5] or [8].

A system $A_n(\xi(x))$ is transformed into a system $B_n(\eta(t))$ by a change of the independent variable $x \mapsto t = \varphi(x)$, if for each solution $y : x \mapsto y(x)$ of $A_n(\xi(x))$ the function $z : t \mapsto z(t) = y(x) = y(\varphi^{-1}(t))$ is a solution of $B_n(\eta(t))$. Here φ^{-1} denotes the inverse to φ; φ^k is the k-th iterate of φ for k positive, and $(-k)$-th iterate of φ^{-1} for k negative; $\varphi^0 = \mathrm{id}$. A system with a deviating argument of the form $x \mapsto x + c$, where $c \neq 0$ is a constant, will be called a *system with a constant deviation.*

We shall prove the following

Theorem 1. *Let* $c \in \mathbb{R}$ *be a constant satisfying* $\operatorname{sign} c = \operatorname{sign}(\xi(x) - x)$. *Any differential system* $A_n(\xi(x))$ *on* I *can be transformed by a change of the independent variable* $x \mapsto t = \varphi(x) \in C_n(I)$, $\varphi'(x) > 0$ *on* I, *into a differential system* $B_n(t + c)$ *on* $J = \varphi(I)$ *with a constant deviation, where* $\varphi(b-) = \infty$. *If the system* $A_n(\xi(x))$ *is linear (with respect to the dependent variable and all its derivatives at* x *and* $\xi(x)$*), then the transformed system* $B_n(t + c)$ *is also linear.*

Proof. Let $y(x)$ be a solution of the system $A_n(\xi(x))$. For a change of the independent variable $x \mapsto t = \varphi(x)$, the function $z(t) = z\,\varphi(x) = y(x)$ is a solution of a system $B_n(\eta(t))$ with a deviating argument η. Since any solution y at $\xi(x)$ should be transformed into a solution z at $\eta(t)$, i.e. $y(\xi(x)) = z(\eta(t))$, or $z\varphi\ \xi(x) = z\eta\ \xi(x)$, we put

$$\varphi\ \xi(x) = \eta\ \varphi(x)\,. \tag{1}$$

Using (1) we can always express the k-th derivative of y at $\xi(x)$ in terms of derivatives of z at $\eta(t)$ of orders $\leqq k$. This follows from the fact that

$$y'(\xi(x)) = \dot{z}\varphi\ \xi(x)\,.\,\varphi'\ \xi(x) = \dot{z}\varphi\xi\ \varphi^{-1}(t)\,.\,\varphi'\xi\ \varphi^{-1}(t) =$$
$$= \dot{z}(\eta(t))\,.\,\varphi'\xi\ \varphi^{-1}(t)\,,$$
$$y''(\xi(x)) = \ddot{z}\ \eta(t)\,.\,\varphi'^2\ \xi(x) + \dot{z}(\eta(t))\,.\,\varphi''\xi\ \varphi^{-1}(t)\,,$$

and in general

$y^{(k)}(\xi(x))$ is a linear combination of $z^{(k)}(\eta(t)),\ z^{(k-1)}(\eta(t)), \ldots, z(\eta(t))$

with coefficients depending on t.

In these expressions the highest degree of the derivatives of φ is equal to $k \leqq n$. This also ensures the linearity of $B_n(\eta(t))$ provided the system $A_n(\xi(x))$ was linear.

For $\eta(t) = t + c$, the relation (1) becomes

$$\varphi(\xi(x)) = \varphi(x) + c\,. \tag{2}$$

First let us consider the case $\xi(x) > x$. Due to Choczewski [4] (see also Kuczma [6, p. 87]), (2) has a solution of the class $C_n(a, b)$. It depends on an arbitrary function defined on any interval of the form $[x_0, \xi(x)]$ and satisfying certain boundary conditions at x_0 and at $\xi(x_0)$. Moreover, if $c > 0$, in accordance with Barvínek [2], there exists a solution $\varphi \in C_n(a, b)$ whose derivative is positive: $\varphi'(x) > 0$ on (a, b).

Under our conditions on ξ, iterations of all positive orders of ξ exist and $\lim_{n\to\infty} \xi^n(x_0) = b$ for any $x_0 \in (a, b)$, see [6, p. 21]. Since $\varphi\ \xi^n(x_0) = \varphi(x_0) + nc$, we have $\lim_{n\to\infty} \varphi\xi^n(x_0) = \varphi(b-) = \infty$.

It remains to consider the case $\xi(x) < x$. In this situation $\xi^{-1}(x) > x$, and the relation (2) can be rewritten as

$$\varphi(\xi^{-1}(u)) = \varphi(u) - c \tag{3}$$

for $u = \xi(x)$. We again use the results of Choczewski, Kuczma, and Barvínek to ensure the existence of a solution φ defined on $(\xi(a_1), b) = (a, b)$, being of the class C_n here. Moreover, if $c < 0$, then there exists a solution φ of (3) that in addition to the above conditions satisfies also $\varphi'(x) > 0$ on (a, b) and $\varphi(b-) = \infty$.

Summarizing, we have constructed a function $\varphi \in C_n(a, b)$, $\varphi'(x) > 0$ on (a, b), $\varphi : (a, b) \to^{\text{onto}} (\varphi(a), \infty)$, satisfying (2). This function considered as a change of the independent variable $x \mapsto t = \varphi(x)$ transforms the differential system $A_n(\xi(x))$ with a deviating argument ξ and defined on (a, b) into a differential system $B_n(t + c)$ with the deviating argument $t + c$ and defined on $(\varphi(a), \infty)$. Q.E.D.

Example. Consider

$$y'(x) = \gamma\, y(x^\alpha)\,, \tag{4}$$

where $\gamma \neq 0$, $\alpha > 0$, $\alpha \neq 1$, $x \in (1, \infty)$; see, e.g., [7]. In our notation $\xi(x) = x^\alpha$. For $\alpha \in (0, 1)$ we have $\xi(x) < x$, and $\alpha \in (1, \infty)$ implies $\xi(x) > x$. Hence $\operatorname{sign}(\xi(x) - x) = \operatorname{sign}(\ln \alpha)$. The relation (2) reads

$$\varphi(x^\alpha) = \varphi(x) + c\,, \tag{5}$$

where $\operatorname{sign} c = \operatorname{sign}(\ln \alpha)$. For $\varphi(x) = \beta \,.\, \ln \ln x$ we have

$$\beta \,.\, \ln(\alpha \ln x) = \beta \,.\, \ln \ln x + c\,, \quad \text{or} \quad \beta = \frac{c}{\ln \alpha}\,.$$

Hence (5) is satisfied for $\varphi(x) = c/\ln \alpha \,.\, \ln \ln x$. Put $t = \varphi(x)$, $y(x) = z(t) = z\,\varphi(x)$. Then $y(x^\alpha) = y\varphi^{-1}\,\varphi(x^\alpha) = z(\varphi(x) + c) = z(t + c)$, and $y'(x) = \mathrm{d}z(t)/\mathrm{d}t \,.$ $.\, \mathrm{d}\varphi(x)/\mathrm{d}x = \dot{z}(t) \,.\, (\mathrm{d}\varphi^{-1}(t)/\mathrm{d}t)^{-1} = \dot{z}(t) \,.\, \exp(\exp(\ln \alpha/c) \,.\, t)) \,.\, \exp((\ln \alpha/c) \,.\, t) \,.$ $.\, (\ln \alpha/c)$. The equation (4) becomes

$$\dot{z}(t) = f(t) \,.\, z(t + c)\,,$$

where $f(t) = \gamma/\beta \,.\, \exp(\exp(t/\beta)) \,.\, \exp(t/\beta)$, $\beta = c/\ln \alpha$.

II. Let a differential system involve several deviating arguments, say $\xi_1, \ldots, \xi_k$ $(k \geqq 2)$. The problem of transformation of the system by a change of the independent variable into a system with deviating arguments $t + c_i$ $(1 \leqq i \leqq k)$ leads to a simultaneous solution φ of k functional equations

$$\varphi\, \xi_i(x) = \varphi(x) + c_i\,, \quad i = 1, \ldots, k\,. \tag{6}$$

In terms of continuous iterations (see Aczél [1] and Kuczma [6]), an equivalent formulation asks for conditions under which a function F exists, satisfying the so called *Translation Equation*

$$F(F(x, u), v) = F(x, u + v)$$

for which $F(x, c_i) = \xi_i(x)$.

Another formulation of the same problem is the following: When can all f_i's $(1 \leqq i \leqq k)$ be extended into a one-parameter continuous group of transformations of a line whose conjugator is of the class C_n? Cf. Borůvka's treatise on the one-parameter continuous group of transformations [3].

To this problem we can give some necessary conditions in

Theorem 2. *If there exists a solution $\varphi \in C_1$, $\varphi' \neq 0$, of a system of functional equations (6) with ξ_i, $1 \leqq i \leqq k$, then each ξ_i and ξ_j commute, and for any (positive, negative, or 0) integers r_i and s_i either $\xi_1^{r_1}\xi_2^{r_2} \ldots \xi_k^{r_k} \equiv \xi_1^{s_1}\xi_2^{s_2} \ldots \xi_k^{s_k}$ or $\xi_1^{r_1}\xi_2^{r_2} \ldots$ $\ldots \xi_k^{r_k}(x_0) \neq \xi_1^{s_1}\xi_2^{s_2} \ldots \xi_k^{s_k}(x_0)$ for each x_0 where the expression is defined.*

Proof. Since $\xi_i = \varphi^{-1}(\varphi(x) + c_i)$ and $\xi_j = \varphi^{-1}(\varphi(x) + c_j)$, we have $\xi_i\xi_j = \varphi^{-1}(\varphi(x) + c_i + c_j) = \xi_j\xi_i$. If $\xi_1^{r_1}\xi_2^{r_2} \dots \xi_k^{r_k}(x_0) = \xi_1^{s_1}\xi_2^{s_2} \dots \xi_k^{s_k}(x_0)$, then $\varphi^{-1}(\varphi(x_0) + \sum_{i=1}^{k} r_i c_i) = \varphi^{-1}(\varphi(x_0) + \sum_{i=1}^{k} s_i c_i)$, or $\sum_{i=1}^{k} r_i c_i = \sum_{i=1}^{k} s_i c_i$.

Hence $\varphi^{-1}(\varphi(x) + \sum_{i=1}^{k} r_i c_i) = \varphi^{-1}(\varphi(x) + \sum_{i=1}^{k} s_i c_i)$, or $\xi_1^{r_1}\xi_2^{r_2} \dots \xi_k^{r_k} \equiv \xi_1^{s_1}\xi_2^{s_2} \dots \xi_k^{s_k}$.

Q.E.D.

Transformations of several deviating arguments were considered by Melvin [7] who used a little different approach, introducing the notion of compatibility of a system of k functions $\xi_1, \dots, \xi_k$ with respect to φ if $\varphi(x) = \mathcal{O}(\varphi(\xi_i(x)))$ as $x \to \infty$ for $i = 1, \dots, k$.

Transformations of linear functional differential equations are considered also in [9], where the form of the most general transformation that converts any linear functional differential equation of the first order into an equation of the same form is derived.

References

[1] *Aczél, J.:* Lectures on Functional Equations and Their Applications. Acad. Press, New York 1966.

[2] *Barvínek, E.:* O rozložení nulových bodů řešení lineární diferenciální rovnice $y'' = Q(t)\,y$ a jejich derivací. Acta Fac. Nat. Univ. Comenian. *5* (1961), 465—474.

[3] *Borůvka, O.:* One-parameter continuous groups of transformations on line. To appear.

[4] *Choczewski, B.:* On differentiable solutions of a functional equation. Ann. Polon. Math. *13* (1963), 133—138.

[5] *Hale, J.:* Functional Differential Equations. Springer, Berlin 1971.

[6] *Kuczma, M.:* Functional Equations in a Single Variable. PWN, Warszawa 1968.

[7] *Melvin, L. H.:* A change of variables for functional differential equations. J. Diff. Equations *18* (1975), 1—10.

[8] *Myškis, A. D.:* Linejnyje Differencial'nyje Uravnenija s Zapazdyvajuščim Argumentom. Nauka, Moskva 1972.

[9] *Tryhuk, V.:* The most general transformation of homogeneous linear retarded differential equation of the first order. To appear.

Author's address: 662 95 Brno, Janáčkovo nám. 2a, ČSSR (Matematický ústav ČSAV).