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ORDER COMPLETIONS OF SEMIPRIME RINGS

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1. INTRODUCTION

Conrad has recently shown that a semiprime ring R is partially ordered by the relation $a \leqq b \Leftrightarrow ara = arb$ for all $r \in R$. This coincides with Abian's order on reduced rings.

Conrad's work was prompted by considerations from the theory of lattice ordered groups. Our approach reflects our background in rings of quotients. Both approaches show that a semiprime ring can be embedded into an orthogonally complete ring S which is a right and left rational extension of R. The ring S has, in fact, a stronger form of completeness and, for a broad class of rings, the ring S has the property that each element of S is the supremum of some orthogonal set in R. In this case S turns out to be the ring of quotients with respect to an idempotent topologizing family of dense right ideals from R.

Later in the article we return to an early paper of Utumi on rings of quotients. When suitably interpreted his work can be used to show that the family of dense right ideals used for Conrad's completion yields the complete ring of right (and left) quotients for an important class of rings, the semisimple WR rings. As a result the Brainerd-Lambek theorems showing that the complete ring of quotients of a Boolean ring is its order completion extend to semisimple WR rings where again the complete ring of right quotients is the order completion, in Conrad's order.

2. PRELIMINARIES

Throughout the article R will denote a semiprime ring with unity. Conrad's relation $a \leqq b$ means that when R is expressed as a subdirect product of prime rings, a and b coincide on the support of a. Consequently the relation is trivial for prime rings. All references to an order relation are to Conrad's order. The terminology is generally that of [9].

The following facts are immediate.

Lemma 1. (i) *If $a \leqq b$ then $ac \leqq bc$ and $ca \leqq cb$ for all $c \in R$.* (ii) *If $arb = ara$ for all $r \in R$ then $bra = ara$ for all $r \in R$.*

A subset X of R is *orthogonal* if for $a, b \in X$, $a \neq b$, $aRb = 0$. A subset X is *boundable* if $aRbR(a - b) = 0$ for all $a, b \in X$. Orthogonal sets are boundable and so are any sets which have upper bounds in R. If X is a subset of R let X^* denote the right annihilator of RXR (which is also the left annihilator). We write a^* for $\{a\}^*$. Boundable sets and suprema of boundable sets are studied in [5] and [8].

Some of the following are used below and others are included because they elucidate suprema.

Lemma 2. (i) *For a subset X and an element a of R, $a = \sup X$ iff a is an upper bound of X and $X^* \subseteq a^*$.* (ii) *If $a = \sup X$ then for all $b \in R$, $\sup bX = ba$ and $\sup Xb = ab$.* (iii) *If X is a set of idempotents and $a = \sup X$ then $a = a^2$. Also if $a = \sup X$ and $a = a^2$ then X consists of idempotents.* (iv) *If $a = \sup X$ then $a \in C$, the centre of R, iff $X \subseteq C$.* (v) *If $X \subseteq C$ and $a = \sup X$ exists in R then a is also the supremum of X in C.*

Proof. (i) and (ii) were established in [5]. (iii) For all $x \in X$, $r \in R$, $xr(a - a^2) = xrx - xrx^2 = 0$. Hence $a + (a - a^2)$ is an upper bound for X. Thus $a \leqq a + (a - a^2)$ giving $aR(a - a^2) = 0$ and from this $(a - a^2)R(a - a^2) = 0$. It follows that $a = a^2$. The converse is also straightforward. (iv) Let $a = \sup X$, $X \subset C$. Then for all $r \in R$, $s \in R$, $x \in X$, $xr(as - sa) = 0$. Hence $a + (as - sa)$ is an upper bound for X. Therefore $a \leqq a + as - sa$ giving, for all $r \in R$, $ar(a + as - sa) = ara$. It follows that $aR(as - sa) = 0$ and that $as = sa$. The converse is immediate, as is (v).

It is interesting to consider the behavior of the partial order under ring extensions. In contrast with the reduced case, the order is not preserved by general extensions of semprime rings. For example if F is a field, the embedding of $F \times F$ into $M_2(F)$ via the diagonal does not preserve the order: $(1, 0) \leqq (1, 1)$ in $F \times F$ but $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not related to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_2(F)$. However, if $R \subseteq S$ are semiprime rings and S_R or ${}_RS$ is an essential extension of R (as modules) then the order is preserved by the extension ([8, II, 2.5]). Further if S a left or right rational extension of R then a boundable set in R is also boundable in S ([6, Lemma 7 (ii)]).

We have called a ring *orthogonally complete* if all orthogonal sets have suprema (*laterally complete* in [7]). Conrad showed that every ring R can be embedded in an orthogonally complete ring which is a right and left rational extension over R. We shall return to this embedding in the next section. We recall that a ring is *complete* if every boundable set has a supremum. This says, in effect, that every set in R which could be bounded above in any order-preserving ring extension already has a supremum in R.

Definition 3. A ring R is *weakly Baer* if every two-sided annihilator ideal is generated by a central idempotent; R is *weakly i-dense* if for any subset X of R, if $X^* \neq 0$ then X^* contains a non-zero central idempotent.

Smith (see [11, p. 108]) has shown that annihilator ideals in semiprime group algebras are generated by their central idempotent so such rings are weakly i-dense. Some commutative examples are found in [2].

Also, if $\{R_i\}_A$ is a family of prime rings then any subring of $\prod_A R_i$ containing $\oplus_A R_i$ is weakly i-dense.

A weakly Baer ring is obviously weakly i-dense but the converse is false. However in [6, Theorem 9] we showed that an orthogonally complete weakly i-dense ring is complete and it readily follows that such rings are weakly Baer, as we now show.

Lemma 4. *An orthogonally complete weakly i-dense ring is weakly Baer.*

Proof. Let X be a subset of R with $X^* \neq 0$. Let $B(R)$ denote the set of central idempotents of R and $Y = \{e \in B(R) \mid e \in X^*\} \neq 0$ (by hypothesis). Now Y is boundable and since, by the result quoted above, R is complete, Y has a supremum f which, by Lemma 2, is in $B(R)$. We claim that $X^* = fR$. Again by Lemma 2 $f^* \supseteq Y^*$ giving $f^* \supseteq Y^* \supseteq X^{**} \supseteq X$ and $f \in X^*$. Consider $X \cup Y$. If $a \in X^*$, $(1 - f)\, a \in (X \cup Y)^*$ and if for some $a \in X^*$, $(1 - f)\, a \neq 0$ we have some $0 \neq g \in (X \cup Y)^* \cap B(R)$. But, $gX = 0$ giving $g \in Y$ and thus $g^2 = 0$, a contradiction. Hence $a = fa$ for all $a \in X^*$.

We also note without proof the following which can be readily generalized from the reduced case.

Proposition 5 (Cf [4, Prop. 16]). *A weakly i-dense semiprime ring is a direct product of prime rings iff it is orthogonally complete and its Boolean algebra of central idempotents is atomic.*

3. CONRAD'S ORTHOGONALLY COMPLETE HULL REVISITED

In his paper [7], Conrad shows how to construct an orthogonally complete weakly Baer extension of an arbitrary semiprime ring. We now study such extensions, obtaining them as rings of right and left quotients related to a certain idempotent topologizing family of right (left) ideals. In the weakly i-dense case it is shown that the resulting ring is an orthogonal completion; i.e., the extension is orthogonally complete (in fact, complete) and every element of it is the supremum of an orthogonal set in the original ring.

In what follows, a subset X of R is *dense* if $X^* = 0$. Let $\mathscr{I}$ be the family of dense ideals of R which are direct sums of annihilator ideals. Now a finite intersection of elements from $\mathscr{I}$ contains an element of $\mathscr{I}$ and so $\mathscr{I}$ is a filter base. Let

$$T = \lim_{I \in \mathscr{I}} \operatorname{Hom}_R(I, R)$$

where the homomorphisms are as right R-modules and the maps of the limit diagram are restrictions. Normally, to ensure that T have a ring structure, it is necessary to impose an extra condition on the filter base $\mathscr{I}$ (i.e., to make the resulting filter "idempotent") but in our case T has a natural ring structure based on composition of homomorphisms since for an annihilator ideal I and $f \in \operatorname{Hom}_R(I, R)$, $f(I) \subseteq I$. Thus multiplication can be defined in the customary way ([9, p. 98]). Further since $\mathscr{I}$ consists of dense ideals, T is a ring of right quotients of R. It is not clear if T is orthogonally complete although an orthogonally complete subring can be constructed as follows. For $I \in \mathscr{I}$ let $\operatorname{Hom}^* I$ be the subset of $\operatorname{Hom}_R(I, R)$ consisting of those homomorphisms $\phi : I \to R$ such that there are annihilator ideals A_α, $\alpha \in \Lambda$, and $r_\alpha \in R$, $\alpha \in \Lambda$ such that $I = \oplus_\Lambda A_\alpha$ and for each α, $\phi(a) = r_\alpha a$ for all $a \in A_\alpha$. This formulation is necessary since I could be expressed as a direct sum of annihilator ideals in several ways. Put $S = \lim_{I \in \mathscr{I}} \operatorname{Hom}^* I$. For $I, J \in \mathscr{I}$, $I = \oplus_\Lambda A_\alpha$, $J = \oplus_\Gamma B_\beta$, A_α, B_β annihilator ideals and $\phi : I \to R$ given by $\phi(a) = r_\alpha a$ for $a \in A_\alpha$ and $\psi : J \to R$ given by $\psi(b) = s_\beta b$ for $b \in B_\beta$ consider $K \in \mathscr{I}$, $K = \oplus(A_\alpha \cap B_\beta)$. Now $\phi + \psi$ and $\phi \cdot \psi$ are both defined on K in the obvious way. These operations induce a ring structure on S.

Proposition 6 (Conrad [7, Th. C]). *Let R be a semiprime ring. Then R has a right and left rational extension S which is weakly Baer and orthogonally complete.*

Proof. Choose S to be the ring constructed above.

(i) S is a left and right rational extension of R. By construction S is a right rational extension. Suppose that s, $t \in S$, $s \neq 0$ with representatives σ, τ, respectively, in $\operatorname{Hom}^* I$ where $I = \oplus A_\alpha$, the A_α annihilator ideals and elements u_α, $v_\alpha \in R$ such that for all $a \in A_\alpha$, $\sigma(a) = u_\alpha a$ and $\tau(a) = v_\alpha a$. For some β, $\sigma A_\beta = u_\beta A_\beta \neq 0$. Then $A_\beta u_\beta A_\beta \neq 0$, so for some $x \in A_\beta$, $xs \neq 0$. Also, $xt \in R$ since $(xt - xv_\beta)(\oplus A_\alpha) = 0$ implies that $xt = xv_\beta \in R$. This shows that S is a left rational extension of R.

(ii) S is orthogonally complete. Let $X = \{s_i\}_\Delta$ be an orthogonal set from S. For each s_i we pick a representative $\phi_i \in \operatorname{Hom}^* \oplus_{\Lambda_i} A_{i\alpha}$ with $\phi_i(a) = r_{i\alpha} a$ for all $a \in A_{i\alpha}$. Put $B_{i\alpha} = (r_{i\alpha} A_{i\alpha})^{**}$, clearly $B_{i\alpha} \subseteq A_{i\alpha}$. Consider $\sum_{i,\alpha} B_{i\alpha}$. Note that for $i \neq j$, $r_{i\alpha} A_{i\alpha} r_{j\beta} A_{j\beta} \subseteq s_i A_{i\alpha} s_j A_{j\beta} = 0$. Hence $B_{i\alpha} B_{j\beta} = 0$. Also, for $\alpha \neq \beta$, $B_{i\alpha} B_{i\beta} = 0$. Since R is semiprime it follows that the sum $\sum_{i,\alpha} B_{i\alpha}$ is direct. Let $C = (\sum_{i,\alpha} B_{i\alpha})^*$ then $J = \sum_{i\alpha} B_{i\alpha} \oplus C \in \mathscr{I}$. Define $\phi \in \operatorname{Hom}^* J$ by $\phi(C) = 0$ and for $b \in B_{i\alpha}$, $\phi(b) = r_{i\alpha} b$. Let $s \in S$ be the element represented by ϕ then we claim that $s = \sup X$ (in S).

For $t \in S$, let $\sigma, \tau, \theta_i \in \operatorname{Hom}^* I$ be representatives of s, t, s_i, respectively, where $I = \oplus_{\Lambda_{i\alpha}} C_{i\alpha\nu} \oplus_{\Gamma_i} C_\beta$, $C_{i\alpha\nu} \subseteq B_{i\alpha}$, the $C_\beta \subseteq C$ and such that there are elements $u_{i\alpha\nu}, v_{i\alpha\nu}, w_{i\alpha\nu}, u_\beta, v_\beta, w_\beta \in R$ giving the action of the respective elements on the $C_{i\alpha\nu}$

and C_β. Calculate $\sigma\tau\sigma_i(c)$ for $c \in C_{i\alpha\nu}$ and $c \in C_\beta$. This shows readily that $\sigma . \tau . \sigma_i = \sigma_i . \tau . \sigma_i$ and that $sts_i = s_its_i$. Thus S is an upper bound of X.

A similar calculation will show that $X^* \subseteq s^*$ (annihilators in S) giving, by Lemma 2, that $s = \sup S$.

(iii) S is weakly Baer. By Lemma 4 it suffices to show that S is weakly i-dense. Let $X \subset S$ with $X^* \neq 0$. Suppose that for $0 \neq s \in S$, $sSX = 0$. Represent s by $\phi : \oplus A_\alpha \to R$, $\phi(a) = r_\alpha a$, for all $a \in A_\alpha$, in the usual fashion. For some α, $sA_\alpha = r_\alpha A_\alpha \neq 0$ but $r_\alpha A_\alpha X = 0$. Put $U = (r_\alpha A_\alpha)^{**}$ and define $\varepsilon : U \oplus U^* \to R$ by $\varepsilon(u) = u$ for $u \in U$, $\varepsilon(u) = 0$ for $u \in U^*$. Then ε represents a central idempotent $e \in S$. Note that for any $t \in S$, $x \in X$ and $r \in R$ with $txr \in R$ we have $r_\alpha A_\alpha txr = 0$ so that $txr \in U^*$. For $t \in S$, $x \in X$ pick representatives δ, τ, ζ of e, t, x respectively defined on $\oplus_A B_\beta \oplus_\Gamma C_\gamma$ with the $B_\beta \subseteq U$ and the $C_\gamma \subseteq U^*$ with actions given by right multiplications by elements of R. One computes directly that $\delta . \tau . \zeta = 0$. Hence $e \in X^*$.

We now sharpen these results in the case R is weakly i-dense. More generally, let $\mathscr{E}$ be the set of right ideals of R which contain a dense set of central idempotents. In [3, Lemma 17] it is shown that $\mathscr{E}$ is an idempotent topologizing filter (or *topology* [13, p. 12]). If R is weakly i-dense the set of dense ideals of the form $\oplus e_\alpha R$, $e_\alpha \in B(R)$ (the set of central idempotents) is cofinal in the filter determined by $\mathscr{I}$. Indeed if $I = \oplus_A A_\alpha$ is a dense direct sum of annihilator ideals, let E_α be a maximal orthogonal set from $A_\alpha \cap B(R)$; then $\oplus E_\alpha R$ is seen to be dense as follows: If not there is $0 \neq e \in (\oplus E_\alpha R)^* \cap B(R)$, but for some β, $eA_\beta \neq 0$ and eA_β, being an annihilator ideal, contains some $0 \neq f \in B(R)$, contradicting the maximality of E_β. Furthermore $\mathrm{Hom}^* (\oplus e_\alpha R) = \mathrm{Hom}_R (\oplus e_\alpha R, R)$.

Proposition 7. *Let R be a semiprime weakly i-dense ring and let S be the ring constructed for Proposition 7. Then $S = Q_{\mathscr{E}}$, the ring of right quotients with respect to $\mathscr{E}$. Further, every element of S is the supremum of an orthogonal set from R.*

Proof. The above remarks show that $Q_{\mathscr{E}} = S$. To establish the second assertion let $s \in S$. Choose a representation $\phi : \oplus_A e_\alpha R \to R$, $\oplus e_\alpha R$ dense, $e_\alpha \in B(R)$ with $(e_\alpha) = r_\alpha = r_\alpha e_\alpha$. Let $X = \{r_\alpha\}$. For $t \in S$ choose a representative $\psi : \oplus_\Gamma f_{\alpha\beta} R \to R$ with each $f_{\alpha\beta} \in B(R)$, $f_{\alpha\beta} R \subseteq e_\alpha R$. Thus ϕ can be restricted to $\oplus f_{\alpha\beta} R$. Now $r_\alpha tsf_{\alpha\beta} = r_\alpha \psi . \phi(f_{\alpha\beta}) = r_\alpha \psi(f_{\alpha\beta} r_\alpha) = r_\alpha \psi(f_{\alpha\beta})\, r_\alpha = r_\alpha tr_\alpha f_{\alpha\beta}$. Hence $r_\alpha ts = r_\alpha tr_\alpha$ for all $t \in S$. Similarly it can be established that $X^* \subseteq s^*$ (annihilators in S). Thus $s = \sup X$.

Proposition 7 generalizes [4, Th. 13] from the commutative case. It is instructive to indicate what happens in a class of examples. Let $\{R_i\}_A$ be a collection of prime rings and R a subring of $\prod_A R_i$ containing $\oplus_A R_i$. In this case $Q_{\mathscr{E}} = \prod_A R_i$ which is not the complete ring of quotients of R unless each R_i is right rationally complete.

It can be remarked in passing that Proposition 3.2 and 3.3 of [12] on polynomial and power series rings carry over to the general semiprime case with the obvious modifications.

4. COMPLETIONS OF SEMISIMPLE WR RINGS

The question of completeness of regular rings is in general not a tractable one since there are prime regular rings which are not left or right self-injective. Hence the results for strongly regular rings, where the order completion is precisely the complete ring of quotients ([3]), do not carry over to the general case. However, some results of Utumi will allow us to show that, for an important class of regular rings, completions and the complete ring of quotients coincide as do self-injectivity and completeness.

Let us recall some definitions from Utumi [14]. A ring is an *I-ring* if every non-nil right ideal contains a non-zero idempotent (regular rings are I-rings). A semiprimitive reduced I-ring is a *plain* ring. A *matrix ideal* of of a ring is one which is isomorphic, as a ring, to some $M_n(T)$ for some n and plain ring T. Finally a ring R is called *semisimple* WR (weakly reducible) if it is semiprimitive and every non-zero ideal contains a non-zero matrix ideal which is generated by a central idempotent. A class of well-known examples of semisimple WR rings is discussed below.

Utumi calls a set B of central idempotents of R a B-family if: (B1) for $f \in B(R)$, if $f = ef$ for some $e \in B$ then $f \in B$ and (B2) if $0 \neq f \in B(R)$ there exists $0 \neq e \in B$ with $ef = e$. A function $\theta : B \to R$ is an *H-mapping* if for $e, f \in B$, $ef = f$ then $\theta(e) f = \theta(f)$.

Lemma 8. *Let R be an i-dense ring. Then* (i) *a subset B of $B(R)$ is a B-family iff it is dense and if for $f \in B(R)$, $e \in B, fe \in B$*; (ii) *there is a one-to-one correspondence between H-mappings and R-homomorphisms $\phi : J \to R$ where J is an ideal containing a B-family.*

Proof. (i) is straightforward.

(ii) Let J be an ideal containing a B-family B. If $\phi : J \to R$ is an R-homomorphism, its restriction to B is clearly an H-mapping. Conversely suppose that $\theta : B \to R$ is an H-mapping. Let S be a maximal orthogonal subset of B. If $0 \neq f \in B(R)$ then $fS \neq 0$; indeed by (B2) $ef = e$ for some $0 \neq e \in B$ and, by maximality, $eS \neq 0$. Define an R-homomorphism $\psi : SR \to R$ by $\psi(e) = \theta(e)$ for $e \in S$. This is well-defined. Further SR is dense since R is i-dense. Finally SR contains the B-family $SR \cap B(R)$.

Utumi ([14, 6.2]) shows that in a semisimple WR ring the complete ring of right quotients can be identified with the ring of equivalence classes of H-mappings. The key to this is the observation that in a semisimple WR ring every dense right ideal (dense in the sense of rings of quotients [9, p. 96]) contains a non-zero central idempotent. It follows from this that in such rings the left and right complete rings of quotients coincide. Lemma 8 then gives the following.

Proposition 9. *Let R be a semiprime weakly i-dense ring. Then $Q_{\mathscr{E}}(R)$ coincides with the ring of equivalence classes of H-mappings as constructed by Utumi.*

Proposition 10. *Let R be a semisimple* WR *ring. Then the complete ring of right quotients, the complete ring of left quotients and $Q_{\mathscr{E}}(R)$ coincide.*

Propositions 10 and 11 represent a very general form of the theorem of Brainerd and Lambek ([1]) on Boolean rings which says that for such a ring R, the complete ring of quotients $Q(R)$ is the order completion of R (in fact, the Dedekind-MacNeille completion).

Proposition 11. *Let R be a semisimple WR ring. The following are equivalent:* (i) *R is orthogonally complete,* (ii) *R is complete,* (iii) *R is right self-injective,* (iv) *R is left self-injective. Further the complete ring of* (*left and right*) *quotients of R is the orthogonal completion of R.*

The most important class of examples of semisimple WR rings is the class of regular rings whose primitive images are all right Artinian (see [10]).

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