Czechoslovak Mathematical Journal

Irena Rachůnková

On a Kneser problem for a system of nonlinear ordinary differential equations

Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 1, 114–126

Persistent URL: http://dml.cz/dmlcz/101730

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

ON A KNESER PROBLEM FOR A SYSTEM OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

IRENA RACHŮNKOVÁ, Olomouc

(Received September 27, 1979)

This paper deals with a problem regarding the existence of a solution of the differential system

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n) \tag{0.1}$$

defined in $[0, +\infty[$ and satisfying the conditions

$$\varphi(x_1(0), \ldots, x_n(0)) = 0\,, \quad x_i(t) \geqq 0 \quad \text{for} \quad t \geqq 0 \quad (i = 1, \ldots, n)\,. \tag{0.2}$$

Problems of such a type for differential equations of the 2nd and higher orders have been studied in [1–5]. Let us recall at this point the works [6], [7] and [8] dealing with analogous problems for differential systems. Unlike [6–8], the existence theorems for (0.1), (0.2) proved in this paper refer to the case when the functions $f_1, \ldots, f_n$ change their signs.

1. FORMULATION OF THE EXISTENCE THEOREMS

In what follows

$$\mathbb{R} =]-\infty, +\infty[\,, \quad \mathbb{R}_+ = [0, +\infty[\,;$$

$L(I)$ is the set of real functions Lebesgue integrable on I;
$L_{loc}(I)$ is the set of real functions Lebesgue integrable on each compact interval contained in I;

$$D_n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}_+ \quad (i = 1, \ldots, n)\}\,;$$

$$D_{nm}(r) = \{(x_1, \ldots, x_n) \in D_n : x_i \leqq r \quad (i = 1, \ldots, m)\}\,.$$

By writing

$$f \in K_{loc}(\mathbb{R}_+ \times D_n)$$

we indicate that the function $f : \mathbb{R}_+ \times D_n \to \mathbb{R}$ satisfies the local Carathéodory conditions, i.e.

$$f(\cdot, x_1, \ldots, x_n) : \mathbb{R}_+ \to \mathbb{R} \quad \text{is measurable for arbitrary} \quad (x_1, \ldots, x_n) \in D_n\,,$$

$$f(t, \cdot, \ldots, \cdot) : D_n \to \mathbb{R} \quad \text{is continuous for almost all} \quad t \in \mathbb{R}_+$$

and

$$\sup \{|f(\cdot, x_1, \ldots, x_n)| : (x_1, \ldots, x_n) \in D_{nn}(\varrho)\} \in L_{\text{loc}}(\mathbb{R}_+)$$

for arbitrary $\varrho \in \mathbb{R}_+$.

We assume throughout the paper that $n \geqq 2$, $f_i \in K_{\text{loc}}(\mathbb{R}_+ \times D_n)$ $(i = 1, \ldots, n)$ and $\varphi : D_n \to \mathbb{R}$ is a continuous function. We seek solutions of the problem (0.1), (0.2) in the class of the vector functions $(x_1, \ldots, x_n) : \mathbb{R}_+ \to D_n$ which are absolutely continuous on each compact interval contained in $\mathbb{R}_+$. The existence theorems proved below refer to such cases when f_i $(i = 1, \ldots, n)$ satisfy the conditions

$$(1.1) \qquad f_i(t, 0, \ldots, 0) = 0\,, \quad f_i(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \leqq 0$$
$$\text{for each} \quad t \in \mathbb{R}_+\,, \quad (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in D_{n-1} \quad (i = 1, \ldots, n)$$

and φ satisfies the conditions

$$(1.2_m) \qquad \varphi(0, \ldots, 0) < 0\,, \quad \varphi(x_1, \ldots, x_n) > 0 \quad \text{for} \quad (x_1, \ldots, x_n) \in D_n\,, \quad \sum_{i=1}^{m} x_i > r$$

with $r \in]0, +\infty[$ and $m \in \{1, \ldots, n\}$.

Theorem 1.1. *Let $m \in \{1, \ldots, n-1\}$ and let the conditions (1.1) and (1.2_m) be fulfilled. Further, let there exist $a \in]0, +\infty[$ and $m_0 \in \{m, \ldots, n-1\}$ such that on the set $[0, a] \times D_{nm}(r)$ the following inequalities hold:*

$$(1.3) \qquad f_i(t, x_1, \ldots, x_n) \leqq 0 \quad (i = 1, \ldots, m_0)\,,$$

$$(1.4) \qquad \sum_{i=1}^{m_k} f_i(t, x_1, \ldots, x_n) \leqq -\delta(x_{m+k}) \quad (k = 1, \ldots, n-m)\,,$$

$$(1.5) \qquad \sum_{i=m+1}^{n} |f_i(t, x_1, \ldots, x_n)| \leqq h(t) \sum_{i=m+1}^{n} (1 + x_i)$$

and on the set $[a, +\infty[\times D_n$ the indequality

$$(1.6) \qquad \sum_{i=1}^{n} f_i(t, x_1, \ldots, x_n) \leqq h(t) \sum_{i=1}^{n} (1 + x_i)$$

is satisfied, where $m_k = \min\{m_0, m + k - 1\}$, $h \in L_{\text{loc}}(\mathbb{R}_+)$, $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ and

$$(1.7) \qquad \lim_{x \to +\infty} \delta(x) = +\infty\,.$$

Then the problem (0.1), (0.2) is solvable.

Remark. The above theorem generalizes Theorem 1 from [8].

Theorem 1.2. *Let the conditions* (1.1) *and* (1.2_n) *be fulfilled. On the set* $\mathbb{R}_+ \times D_n$ *let the inequality*

$$\sum_{i=1}^{n} f_i(t, x_1, \ldots, x_n) \leqq g(t, \sum_{i=1}^{n} x_i) \tag{1.8}$$

be satisfied, where $g \in K_{\mathrm{loc}}(\mathbb{R}_+ \times D_1)$, *and let the Cauchy problem*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = g(t, u), \quad u(0) = r \tag{1.9}$$

have an upper solution u^* *defined on the whole* $\mathbb{R}_+$. *Then the problem* (0.1), (0.2) *is solvable.*

Corollary. *Let the conditions* (1.1) *and* (1.2_n) *be fulfilled and let on the set* $\mathbb{R}_+ \times D_n$ *the inequality*

$$\sum_{i=1}^{n} f_i(t, x_1, \ldots, x_n) \leqq h(t) \sum_{i=1}^{n} (1 + x_i)$$

be satisfied with $h \in L_{\mathrm{loc}}(R_+)$. *Then the problem* (0.1), (0.2) *is solvable.*

Remark. If f_i $(i = 1, \ldots, n)$ are negative functions and $\varphi(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i$, then the last proposition leads to one theorem due to Hartman-Wintner-Coffman [6, 7].

2. LEMMAS ON A PRIORI ESTIMATES

Lemma 2.1. *Suppose that* $m \in \{1, \ldots, n-1\}$, $m_0 \in \{m, \ldots, n-1\}$, $m_k = \min\{m_0, m + k - 1\}$ $(k = 1, \ldots, n - m)$, $0 < a < +\infty$, $0 < r < +\infty$. *Let* $h \in L_{\mathrm{loc}}(R_+)$ *be a nonnegative function and let* $a_* \in \,]0, a[$ *satisfy*

$$\int_0^{a_*} h(\tau) \, \mathrm{d}\tau < \tfrac{1}{2} \,. \tag{2.1}$$

Let further $\delta_0 : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nondecreasing continuous function satisfying the condition*

$$\lim_{x \to +\infty} \delta_0(x) > \frac{1}{\varepsilon} \sum_{k=0}^{s} r_k \quad (s = 0, 1, 2, \ldots, m_0 - m)\,, \tag{2.2}$$

where

$$\varepsilon = \frac{a_*}{m_0 - m + 1}, \quad r_0 = r\,, \quad r_k = \delta_0^{-1}\left(\frac{r + r_1 + \ldots + r_{k-1}}{\varepsilon}\right)$$

$(k = 1, \ldots, m_0 - m)$ *and* δ_0^{-1} *is the inverse function to* δ_0. *Then there exists* $r^* > r$

such that for arbitrary $b > a$ and for arbitrary absolutely continuous functions $x_i : [0, b] \to \mathbb{R}_+$ $(i = 1, \ldots, n)$ the inequalities

$$\sum_{i=1}^{m} x_i(0) \leqq r\,, \quad x_i(t) \geqq 0 \quad (i = 1, \ldots, n) \quad \text{for} \quad 0 \leqq t \leqq b\,, \tag{2.3}$$

$$x_i'(t) \leqq 0 \quad for \quad 0 \leqq t \leqq a \quad (i = 1, \ldots, m_0)\,, \tag{2.4}$$

$$\sum_{i=1}^{m_k} x_i'(t) \leqq -\delta_0(x_{m+k}(t)) \quad for \quad 0 \leqq t \leqq a \quad (k = 1, \ldots, n - m)\,, \tag{2.5}$$

$$\sum_{i=m+1}^{n} |x_i'(t)| \leqq h(t) \sum_{i=m+1}^{n} (1 + x_i(t)) \quad for \quad 0 \leqq t \leqq a \tag{2.6}$$

and

$$\sum_{i=1}^{n} x_i'(t) \leqq h(t) \sum_{i=1}^{n} (1 + x_i(t)) \quad for \quad a \leqq t \leqq b \tag{2.7}$$

imply the estimate

$$\sum_{i=1}^{n} x_i(t) \leqq r^* \exp\left[\int_0^t h(\tau)\,\mathrm{d}\tau\right] \quad for \quad 0 \leqq t \leqq b\,. \tag{2.8}$$

Proof. By (2.3) and (2.4) it holds that

$$0 \leqq \sum_{i=1}^{m} x_i(t) \leqq r \quad \text{for} \quad 0 \leqq t \leqq a\,. \tag{2.9}$$

From (2.4), (2.5) and (2.9) we get (for $m_0 > m$)

$$r \geqq -\sum_{i=1}^{m} \int_0^{\varepsilon} x_i'(\tau)\,\mathrm{d}\tau \geqq \int_0^{\varepsilon} \delta_0(x_{m+1}(\tau))\,\mathrm{d}\tau \geqq \varepsilon\delta_0(x_{m+1}(\varepsilon))\,.$$

This implies

$$0 \leqq x_{m+1}(t) \leqq x_{m+1}(\varepsilon) \leqq r_1 \quad \text{for} \quad \varepsilon \leqq t \leqq a\,.$$

Similarly (for $m_0 > m + 1$)

$$r + r_1 + \ldots + r_{k-1} \geqq -\sum_{i=1}^{m_k} \int_{(k-1)\varepsilon}^{k\varepsilon} x_i'(\tau)\,\mathrm{d}\tau \geqq \int_{(k-1)\varepsilon}^{k\varepsilon} \delta_0(x_{m+k}(\tau))\,\mathrm{d}\tau \geqq \varepsilon\delta_0(x_{m+k}(k\varepsilon))$$

and

$$0 \leqq x_{m+k}(t) \leqq x_{m+k}(k\varepsilon) \leqq r_k \quad \text{for} \quad k\varepsilon \leqq t \leqq a \quad (k = 2, \ldots, m_0 - m)\,.$$

If we put

$$\varrho_0 = \sum_{k=0}^{m_0 - m} r_k$$

then we get (for $m_0 \geqq m$)

$$0 \leqq \sum_{i=1}^{m_0} x_i(t) \leqq \varrho_0 \quad \text{for} \quad (m_0 - m)\,\varepsilon \leqq t \leqq a\,. \tag{2.10}$$

Multiplying (2.5) (for $k = m_0 + 1 - m, \ldots, n - m$) by -1 and integrating from $(m_0 - m)\,\varepsilon$ to a_* we have in accordance with (2.10)

$$\varrho_0 \geqq \int_{(m_0-m)\varepsilon}^{a_*} \delta_0(x_i(t))\,\mathrm{d}t \quad (i = m_0 + 1, \ldots, n)\,.$$

This implies the existence of points

$$t_i \in [(m_0 - m)\,\varepsilon, a_*] \quad (i = m_0 + 1, \ldots, n)$$

such that

$$0 \leqq x_i(t_i) \leqq \delta_0^{-1}\left(\frac{\varrho_0}{\varepsilon}\right) \quad (i = m_0 + 1, \ldots, n)\,. \tag{2.11}$$

Put

$$t_i = a_* \quad (i = m + 1, \ldots, m_0)\,,$$

$$\varrho_1 = \varrho_0 + n\delta_0^{-1}\left(\frac{\varrho_0}{\varepsilon}\right).$$

Then (2.10) and (2.11) imply

$$\sum_{i=m+1}^{n} x_i(t_i) \leqq \varrho_1\,. \tag{2.12}$$

From (2.6) and (2.12) we obtain

$$\sum_{i=m+1}^{n} x_i(t) \leqq \varrho_1 + \int_0^{a_*} h(\tau) \sum_{i=m+1}^{n} (1 + x_i(\tau))\,\mathrm{d}\tau \quad \text{for} \quad 0 \leqq t \leqq a_*\,.$$

If we denote

$$\varrho^* = \max\left\{\sum_{i=m+1}^{n} (1 + x_i(t)) : 0 \leqq t \leqq a_*\right\}$$

then we get from the last inequality with respect to (2.1) the relation

$$\varrho^* \leqq \varrho_1 + n + \frac{\varrho^*}{2}$$

and thus

$$\sum_{i=m+1}^{n} (1 + x_i(t)) \leqq \varrho^* \leqq 2(\varrho_1 + n) \quad \text{for} \quad 0 \leqq t \leqq a_*\,. \tag{2.13}$$

(2.6) and (2.13) imply the inequality

$$\sum_{i=m+1}^{n} (1 + x_i(t)) \leqq 2(\varrho_1 + n) \exp\left[\int_{a_*}^{t} h(\tau)\,\mathrm{d}\tau\right] \quad \text{for} \quad a_* \leqq t \leqq a\,. \tag{2.14}$$

In accordance with (2.9), (2.13) and (2.14) we have

$$\sum_{i=1}^{n} (1 + x_i(t)) \leqq r^* \quad \text{for} \quad 0 \leqq t \leqq a \tag{2.15}$$

(remember that $\varrho_1 > \varrho_0 > r$), where

$$r^* = 3(\varrho_1 + n)\exp\left[\int_0^a h(\tau)\,\mathrm{d}\tau\right].$$

Integrating (2.7) from a to t we get by (2.15)

$$\sum_{i=1}^{n}(1 + x_i(t)) \leqq r^* \exp\left[\int_a^t h(\tau)\,\mathrm{d}\tau\right] \quad \text{for} \quad a \leqq t \leqq b\,.$$

Thus, we can conclude that the estimate (2.8) is valid, where r^* is constant independent both of $(x_1, \ldots, x_n)$ and of b.

By means of Lemma on differential inequalities (see [9], p. 48, Lemma 4.3) we obtain

Lemma 2.2. *Let $g \in K_{\mathrm{loc}}(\mathbb{R}_+ \times D_1)$ and let the problem (1.9) have an upper solution u^* defined on the whole $\mathbb{R}_+$. Then for arbitrary $b > 0$ and for arbitrary absolutely continuous functions $x_i : [0, b] \to \mathbb{R}_+$ $(i = 1, \ldots, n)$ satisfying the inequalities*

$$(2.16) \qquad \sum_{i=1}^{n} x_i(0) \leqq r\,, \quad \sum_{i=1}^{n} x_i'(t) \leqq g(t, \sum_{i=1}^{n} x_i(t)) \quad \textit{for} \quad 0 \leqq t \leqq b$$

the estimate

$$(2.17) \qquad \sum_{i=1}^{n} x_i(t) \leqq u^*(t) \quad \textit{for} \quad 0 \leqq t \leqq b$$

holds.

3. LEMMA ON SOLVABILITY OF A CERTAIN AUXILIARY BOUNDARY VALUE PROBLEM

In what follows we will use the following

Lemma 3.1. *Suppose that $f_{ip} \in K_{\mathrm{loc}}(\mathbb{R}_+ \times D_n)$ $(i = 1, \ldots, n,\ p = 1, 2, \ldots)$ and the following relations are satisfied on the set $\mathbb{R}_+ \times D_n$:*

$$\sum_{i=1}^{n} |f_{ip}(t, x_1, \ldots, x_n)| \leqq f_0(t, x_1, \ldots, x_n) \quad (p = 1, 2, \ldots)$$

and

$$\lim_{p \to +\infty} f_{ip}(t, x_1, \ldots, x_n) = f_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n)$$

where $f_0 \in K_{\mathrm{loc}}(\mathbb{R}_+ \times D_n)$. For each natural p let the differential system

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_{ip}(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n)$$

have a solution $(x_1, \dots, x_n)$ *satisfying* (0.2) *and let* $\psi(t) \in L_{\text{loc}}(\mathbb{R}_+)$ *be such that the inequality*

$$\sup \{ \sum_{i=1}^{n} |x_{ip}(t)| : p = 1, 2, \dots \} \leqq \psi(t) \quad \text{for} \quad t \in \mathbb{R}_+$$

holds. Then the sequence of the vector functions $\{(x_{1p}, \dots, x_{np})\}_{p=1}^{+\infty}$ *contains a uniformly converging subsequence such that its limit is a solution of the problem* (0.1), (0.2).

Proof. See [9], *p.* 43–48.

For the system (0.1) we consider an auxiliary boundary value problem

(3.1) $$\varphi(x_1(0), \dots, x_n(0)) = 0\,, \quad x_i(b) = 0 \quad (i = 2, 3, \dots, n)\,,$$

where $b > 0$.

Lemma 3.2. *Suppose that the conditions* (1.1) *and* (1.2_1) *hold and*

(3.2) $$\sum_{i=1}^{n} |f_i(t, x_1, \dots, x_n)| \leqq h^*(t) \sum_{i=2}^{n} (1 + x_i)$$

on the set $[0, b] \times D_n$, *where* $h^* \in L([0, b])$. *Then the problem* (0.1), (3.1) *has at least one solution* $(x_1, \dots, x_n)$ *such that*

(3.3) $$x_i(t) \geqq 0 \quad \text{for} \quad 0 \leqq t \leqq b \quad (i = 1, \dots, n)\,.$$

Proof. First let us prove Lemma under the additional assumption that the right-hand sides of the system (0.1) satisfy the local Lipschitz conditions with respect to their last n arguments, i.e., for arbitrary $\varrho > 0$ we have

(3.4) $$\sum_{i=1}^{n} |f_i(t, x_1, \dots, x_n) - f_i(t, y_1, \dots, y_n)| \leqq l_\varrho(t) \sum_{i=1}^{n} |x_i - y_i|$$
$$\text{for} \quad 0 \leqq t \leqq b\,, \quad 0 \leqq x_i \leqq \varrho\,, \quad 0 \leqq y_i \leqq \varrho \quad (i = 1, \dots, n)\,,$$

where $l_\varrho \in L([0, b])$.

Put

$$\sigma(s) = \begin{cases} 0 & \text{for} \quad s \leqq 0\,, \\ s & \text{for} \quad s > 0\,, \end{cases}$$

(3.5) $$\tilde{f}_i(t, x_1, \dots, x_n) = f_i(t, \sigma(x_1), \dots, \sigma(x_n)) \quad (i = 1, \dots, n)$$

and consider the system

(3.6) $$\frac{dx_i}{dt} = \tilde{f}_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n)$$

under initial conditions

(3.7) $$x_1(b) = \alpha\,, \quad x_i(b) = 0 \quad (i = 2, \dots, n)\,.$$

According to (3.2) and (3.4), for arbitrary $\alpha \in \mathbb{R}$ the problem (3.6), (3.7) has a unique solution $(x_1(\cdot; \alpha), \ldots, x_n(\cdot; \alpha))$ defined on the whole segment $[0, b]$.

Put

$$k_i(\alpha, x_1, \ldots, x_n) = \tilde{f}_i(t, x_1(t; \alpha), \ldots, x_n(t; \alpha)),$$

$$l_i(t; \alpha) = \begin{cases} \dfrac{k_i(\alpha, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) - k_i(\alpha, x_1, \ldots, x_n)}{x_i(t; \alpha)} & \text{for} \quad x_i(t; \alpha) \neq 0, \\ 0 \quad \text{for} \quad x_i(t; \alpha) = 0 \end{cases}$$

$(i = 1, \ldots, n)$. Using (1.1) and (3.5) we have

$$\frac{\mathrm{d}x_i(t; \alpha)}{\mathrm{d}t} \leqq -l_i(t; \alpha)\, x_i(t; \alpha) \quad \text{for} \quad 0 \leqq t \leqq b \quad (i = 1, \ldots, n).$$

Consequently

$$x_1(t; \alpha) \geqq \alpha \exp\left[\int_t^b l_1(\tau; \alpha)\, \mathrm{d}\tau\right] \geqq 0, \quad x_i(t; \alpha) \geqq 0 \quad \text{for} \quad 0 \leqq t \leqq b, \quad \alpha \geqq 0$$
$$(i = 2, \ldots, n).$$

This implies that the vector function $(x_1(\cdot; \alpha), \ldots, x_n(\cdot; \alpha))$ is a solution of the system (0.1) for arbitrary $\alpha \in \mathbb{R}_+$. On the other hand, the relation (3.2) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=2}^{n} [1 + x_i(t; \alpha)] \leqq -h^*(t) \sum_{i=2}^{n} (1 + x_i(t; \alpha)) \quad \text{for} \quad 0 \leqq t \leqq b, \quad \alpha \geqq 0.$$

Therefore

$$\text{(3.8)} \quad \sum_{i=2}^{n} [1 + x_i(t; \alpha)] \leqq \sum_{i=2}^{n} [1 + x_i(b; \alpha)] \exp\left[\int_t^b h^*(\tau)\, \mathrm{d}\tau\right] \leqq n \exp\left[\int_0^b h^*(\tau)\, \mathrm{d}\tau\right]$$
$$\text{for} \quad 0 \leqq t \leqq b, \quad \alpha \geqq 0.$$

Put

$$\tilde{\varphi}(\alpha) = \varphi(x_1(0; \alpha), \ldots, x_n(0; \alpha)),$$

$$\alpha^* = r + n \int_0^b h^*(\tau)\, \mathrm{d}\tau \,.\, \exp\left[\int_0^b h^*(\tau)\, \mathrm{d}\tau\right].$$

Following (3.2) and (3.8),

$$x_1(0; \alpha^*) = \alpha^* - \int_0^b f_1(\tau, x_1(\tau, \alpha^*), \ldots, x_n(\tau, \alpha^*))\, \mathrm{d}\tau \geqq$$

$$\geqq \alpha^* - \int_0^b h^*(\tau) \sum_{i=2}^{n} [1 + x_i(\tau; \alpha^*)]\, \mathrm{d}\tau \geqq$$

$$\geqq \alpha^* - n \int_0^b h^*(\tau)\, \mathrm{d}\tau \,.\, \exp\left[\int_0^b h^*(\tau)\, \mathrm{d}\tau\right] = r.$$

Thus, it follows from (1.2_1) that

$$\tilde{\varphi}(\alpha^*) \geqq 0 .$$

On the other hand, $\tilde{\varphi}$ is continuous on $[0, \alpha^*]$ and

$$\tilde{\varphi}(0) = \varphi(x_1(0, 0), \ldots, x_n(0, 0)) = \varphi(0, 0, \ldots, 0) < 0 .$$

So there exists $\alpha_0 \in]0, \alpha^*]$ such that

$$\tilde{\varphi}(\alpha_0) = 0 .$$

Obviously, $(x_1(\cdot\,; \alpha_0), \ldots, x_n(\cdot\,; \alpha_0))$ is a solution of the problem (0.1), (3.1) and it satisfies the conditions (3.3). To complete the proof of Lemma we must get rid of the additional assumption (3.4). Let $\tilde{f}_i$ $(i = 1, \ldots, n)$ be the functions given by the identities (3.5) and let $\omega_m : R \to R_+$ $(m = 1, 2, \ldots)$ be the sequence of continuously differentiable functions such that

$$\omega_m(x) = 0 \quad \text{for} \quad |x| \geqq \frac{1}{m}, \quad \int_{-\infty}^{+\infty} \omega_m(x)\, \mathrm{d}x = 1 \quad (m = 1,\ 2, \ldots) .$$

Put

$$g_{im}(t, x_1, \ldots, x_n) = \int_{-\infty}^{+\infty} \omega_m(y_1 - x_1)\, \mathrm{d}y_1 \ldots \int_{-\infty}^{+\infty} \omega_m(y_n - x_n)\, \tilde{f}_i(t, y_1, \ldots, y_n)\, \mathrm{d}y_n$$

$$(i = 1, \ldots, n) ,$$

$$h_{im}(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) =$$

$$= \int_{-\infty}^{+\infty} \omega_m(y_1 - x_1)\mathrm{d}y_1 \ldots \int_{-\infty}^{+\infty} \omega_m(y_{i-1} - x_{i-1})\mathrm{d}y_{i-1} \cdot \int_{-\infty}^{+\infty} \omega_m(y_{i+1} - x_{i+1})\mathrm{d}y_{i+1} \ldots$$

$$\ldots \int_{-\infty}^{+\infty} \omega_m(y_n - x_n)\, \tilde{f}_i(t, y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n)\, \mathrm{d}y_n \quad (i = 1, \ldots, n - 1) ,$$

$$h_{nm}(t, x_1, \ldots, x_{n-1}) =$$

$$= \int_{-\infty}^{+\infty} \omega_m(y_1 - x_1)\, \mathrm{d}y_1 \ldots \int_{-\infty}^{+\infty} \omega_m(y_{n-2} - x_{n-2})\, \mathrm{d}y_{n-2} \cdot \int_{-\infty}^{+\infty} \omega_m(y_{n-1} - x_{n-1}) \,\cdot$$

$$\cdot\, \tilde{f}_n(t, y_1, \ldots, y_{n-1}, 0)\, \mathrm{d}y_{n-1}$$

and

$$f_{im}(t, x_1, \ldots, x_n) = g_{im}(t, x_1, \ldots, x_n) - g_{im}(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) -$$

$$- \left|h_{im}(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - h_{im}(t, 0, \ldots, 0)\right| \quad (i = 1, \ldots, n) .$$

Then

(3.9) $$f_{im}(t, 0, \ldots, 0) = 0 , \quad f_{im}(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \leqq 0$$

$$\text{for} \quad 0 \leqq t \leqq b , \quad (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in R^{n-1}$$

and

$$(3.10) \quad \sum_{i=1}^{n} |f_{im}(t, x_1, \ldots, x_n)| = 4h^*(t) \sum_{i=2}^{n} \left(1 + \frac{1}{m} + |x_i|\right) = 8h^*(t) \sum_{i=2}^{n} (1 + |x_i|)$$

$$\text{for} \quad 0 \leqq t \leqq b\,, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n\,.$$

Moreover, for any $t \in [0, +\infty[$

$$(3.11) \qquad \lim_{m \to +\infty} f_{im}(t, x_1, \ldots, x_n) = \tilde{f}_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n)$$

uniformly on each bounded set of the space $\mathbb{R}^n$. It is obvious from the structure of the functions f_{im} $(i = 1, \ldots, n)$ that these functions satisfy the local Lipschitz conditions with respect to their last n arguments. Thus in accordance with the results proved above, for each natural m the system

$$(3.12) \qquad \frac{dx_i}{dt} = f_{im}(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n)$$

has a solution $(x_{1m}, \ldots, x_{nm})$ which satisfies (3.1) and (3.3). Using (1.2_1), (3.9), (3.10) and (3.11) we can prove that the systems (3.12) and their solutions $(x_{1m}, \ldots, x_{nm})$ satisfy the conditions of Lemma 3.1 and thus the sequence of the vector functions $\{(x_{1m}, \ldots, x_{nm})\}_{m=1}^{+\infty}$ contains a subsequence uniformly convergent on $[0, b]$. The limit of the subsequence is a solution of the problem (0.1), (3.1).

4. PROOFS OF THE EXISTENCE THEOREMS

Proof of Theorem 1.1. Let r^*, ε, r_k $(k = 0, \ldots, m_0 - m)$ be the constants appearing in Lemma 2.1. Choose a number $c_0 \in]r, +\infty[$ and a nondecreasing continuous function $\delta_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\delta(x) \geqq \delta_0(x) \quad \text{for} \quad x \geqq 0$$

and

$$\delta_0(x) = \delta_0(c_0) > \frac{1}{\varepsilon} \sum_{k=0}^{m_0 - m} r_k \quad \text{for} \quad x \geqq c_0\,.$$

Put

$$\varrho(t) = r^* \exp\left[\int_0^t h(\tau)\, d\tau\right] + c_0\,,$$

$$\sigma(t, s) = \begin{cases} s & \text{for} \quad 0 \leqq s \leqq \varrho(t)\,, \\ \varrho(t) & \text{for} \quad s > \varrho(t)\,, \end{cases}$$

$$\tilde{f}_i(t, x_1, \ldots, x_n) = f_i(t, \sigma(t, x_1), \ldots, \sigma(t, x_n)) \quad (i = 1, \ldots, n)$$

and consider the differential system

$$\frac{dx_i}{dt} = \tilde{f}_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n). \tag{4.1}$$

The definition of $\tilde{f}_i$ $(i = 1, \ldots, n)$ together with the conditions (1.3)–(1.6) yields

$$\tilde{f}_i(t, x_1, \ldots, x_n) = f_i(t, x_1, \ldots, x_n) \tag{4.2}$$

$$\text{for} \quad t \geqq 0, \quad \sum_{i=1}^{n} x_i \leqq \varrho(t), \quad x_i \geqq 0 \quad (i = 1, \ldots, n),$$

$$\tilde{f}_i(t, x_1, \ldots, x_n) \leqq 0 \quad (i = 1, \ldots, m_0), \tag{4.3}$$

$$\sum_{i=1}^{m_k} \tilde{f}_i(t, x_1, \ldots, x_n) \leqq -\delta_0(x_{m+k}) \quad (k = 1, \ldots, n - m), \tag{4.4}$$

$$\sum_{i=m+1}^{n} |\tilde{f}_i(t, x_1, \ldots, x_n)| \leqq h(t) \sum_{i=m+1}^{n} (1 + x_i) \tag{4.5}$$

$$\text{for} \quad (t, x_1, \ldots, x_n) \in [0, a] \times D_{nm}(r)$$

and

$$\sum_{i=1}^{n} \tilde{f}_i(t, x_1, \ldots, x_n) = h(t) \sum_{i=1}^{n} (1 + x_i(t)) \tag{4.6}$$

$$\text{for} \quad (t, x_1, \ldots, x_n) \in [a, +\infty[\times D_n.$$

Further

$$\sum_{i=1}^{n} |\tilde{f}_i(t, x_1, \ldots, x_n)| \leqq h^*(t) \quad \text{for} \quad (t, x_1, \ldots, x_n) \in [0, b] \times D_n, \tag{4.7}$$

where

$$h^*(t) = \max \left\{ \sum_{i=1}^{n} |f_i(t, x_1, \ldots, x_n)| : 0 \leqq x_i \leqq \varrho(t), \ (i = 1, \ldots, n) \right\}$$

and $h^* \in L_{\text{loc}}([0, +\infty[)$.

According to Lemma 3.2, for each natural p the system (4.1) has a solution $(x_{1p}, \ldots, x_{np})$ defined on $[0, a + p]$ and satisfying the conditions

$$\varphi(x_{1p}(0), \ldots, x_{np}(0)) = 0, \quad x_{ip}(t) \geqq 0 \tag{4.8}$$

$$\text{for} \quad 0 \leqq t \leqq a + p \quad (i = 1, \ldots, n).$$

(1.3) and (1.2_m) imply

$$\sum_{i=1}^{m} x_{ip}(t) \leqq \sum_{i=1}^{m} x_{ip}(0) \leqq r \quad \text{for} \quad 0 \leqq t \leqq a. \tag{4.9}$$

On the other hand, since (4.3)–(4.6) hold we have

$$x'_{ip}(t) \leqq 0 \quad \text{for} \quad 0 \leqq t \leqq a \quad (i = 1, \ldots, m_0), \tag{4.10}$$

$$(4.11)\qquad \sum_{i=1}^{m_k} x'_{ip}(t) \leqq -\delta_0(x_{m+k,p}(t)) \quad \text{for} \quad 0 \leqq t \leqq a \quad (k = 1, \ldots, n - m),$$

$$(4.12)\qquad \sum_{i=m+1}^{n} |x'_{ip}(t)| \leqq h(t) \sum_{i=m+1}^{n} (1 + x_{ip}(t)) \quad \text{for} \quad 0 \leqq t \leqq a$$

and

$$(4.13)\qquad \sum_{i=1}^{n} x'_{ip}(t) \leqq h(t) \sum_{i=1}^{n} (1 + x_{ip}(t)) \quad \text{for} \quad a \leqq t \leqq a + p\,.$$

On the basis of Lemma 2.1 we get from (4.8)–(4.13) the estimate

$$(4.14)\qquad \sum_{i=1}^{n} x_{ip}(t) \leqq r^* \exp\left[\int_0^t h(\tau)\, d\tau\right] \quad \text{for} \quad 0 \leqq t \leqq a + p\,.$$

We can deduce from (4.2) and (4.14) that $(x_{1p}, \ldots, x_{np})$ is a solution of the system (0.1) on $[0, a + p]$.

Taking (4.7) and (4.14) into consideration we can prove (by Lemma 3.1) that from the sequence of the vector functions $\{(x_{1p}, \ldots, x_{np})\}_{p=1}^{+\infty}$ we can choose a subsequence $\{(x_{1p_m}, \ldots, x_{np_m})\}_{m=1}^{+\infty}$ such that this subsequence converges uniformly on each segment from $[0, +\infty[$ and

$$(x_1, \ldots, x_n) = \lim_{m \to +\infty} (x_{1p_m}, \ldots, x_{np_m})$$

is a solution of the system (0.1) on $[0, +\infty[$. On the other hand, it is obvious from (4.8) that $(x_1, \ldots, x_n)$ satisfies the conditions (0.2).

Proof of theorem 1.2. Put

$$\sigma(t, s) = \begin{cases} 1 & \text{for} \quad 0 \leqq s \leqq u^*(t)\,, \\ 2 - \dfrac{s}{u^*(t)} & \text{for} \quad u^*(t) < s < 2\,u^*(t)\,, \\ 0 & \text{for} \quad s \geqq 2\,u^*(t)\,, \end{cases}$$

$$\tilde{f}_i(t, x_1, \ldots, x_n) = \sigma(t, \sum_{i=1}^{n} x_i) f_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n),$$

where u^* is an upper solution of the problem (1.9) and consider the differential system

$$(4.15)\qquad \frac{dx_i}{dt} = \tilde{f}_i(t, x_1, \ldots, x_n) \quad (i = 1, \ldots, n)\,.$$

The definition of $\tilde{f}_i$ $(i = 1, \ldots, n)$ together with (1.8) implies that

$$(4.16)\qquad \sum_{i=1}^{n} \tilde{f}_i(t, x_1, \ldots, x_n) \leqq g(t, \sum_{i=1}^{n} x_i)$$

on $\mathbb{R}_+ \times D_n$ and

$$\sum_{i=1}^{n} |\tilde{f}_i(t, x_1, \ldots, x_n)| \leqq f^*(t),$$

where

$$f^*(t) = \max \{ \sum_{i=1}^{n} |f_i(t, x_1, \ldots, x_n)| : \sum_{i=1}^{n} x_i \leqq 2\, u^*(t)\}$$

and $f^* \in L_{\mathrm{loc}}([0, +\infty[)$.

According to Lemma 3.2, for each natural p the system (4.15) has a solution $(x_{1p}, \ldots, x_{np})$ defined on $[0, a + p]$ and satisfying (4.8). Further, using (1.2_n) and (4.16) we get

$$(4.17) \qquad \sum_{i=1}^{n} x_{ip}(0) \leqq r\,, \quad \sum_{i=1}^{n} x'_{ip}(t) \leqq g(t. \sum_{i=1}^{n} x_{ip}(t)) \quad \text{for} \quad 0 \leqq t \leqq a + p\,.$$

From (4.17) by Lemma 2.2 we have

$$\sum_{i=1}^{n} x_{ip}(t) \leqq u^*(t) \quad \text{for} \quad 0 \leqq t \leqq a + p\,.$$

The rest of the proof is analogous to that of Theorem 1.1.

References

[1] *A. Kneser:* Untersuchung und asymptotische Darstellung der Integrale gewisser Differentialgleichungen bei grossen reellen Werthen des Arguments, Journ. für die reine und angew. Math., *116* (1896), 178—212.

[2] *A. Mambriani:* Sul un teorema relative alle equazioni differenziali ordinairie del 2[do] ordine, Rend. R. Acc. Naz. dei Lincei, *9* (1929), 620—622.

[3] *P. Hartman, A. Wintner:* On the non-increasing solutions of $y'' = f(x, y, y')$, Amer. Journ. of Math., *73* (1951), 390—404.

[4] *I. T. Kiguradze:* On the non-negative non-increasing solutions of non-linear second order differential equations, Ann. di Mat. pura ed appl., *81* (1969), 169—192.

[5] *I. T. Kiguradze:* O monotonnych rešenijach nělinějnych obyknověnnych differencial'nych uravněnij n-go porjadka, Izv. AN SSSR, ser. matem., *33* (1969), 1373—1398.

[6] *P. Hartman, A. Wintner:* On monotone solutions of systems of non-linear differential equations, Amer. Journ. of Math., *76* (1954), 860—866.

[7] *C. V. Coffman:* Non-linear differential equations on cones in Banach spaces, Pacific Journ. Math., *14* (1964), 9—16.

[8] *T. A. Čanturija:* O zadače tipa Knesera dlja sistemy obyknověnnych differencial'nych uravněnij, Matem. zametki, *15* (1974), 897—906.

[9] *I. T. Kiguradze:* Někotoryje singuljarnyje krajevyje zadači dlja obyknověnnych differencial'nych uravněnij, Izd. Tbilisskogo univ., Tbilisi, 1975.

Author's address: 771 46 Olomouc, Gottwaldova 15, ČSSR (Přírodovědecká fakulta UP).