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# CONGRUENCES ON CONVENTIONAL SEMIGROUPS

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## 1. INTRODUCTION AND SUMMARY

A conventional semigroup $S$ is a regular semigroup in which $E$, the set of idempotents, is self-conjugate; that is $cEc' \subseteq E$ for each $c \in S$ and for each inverse $c'$ of $c$. This property was derived from orthodox semigroups wherein the set of idempotents, being a subsemigroup, is inherently self-conjugate. Conventional semigroups were first developed by Masat in [5] and stemmed directly from generalizations of Meakin's work in [6].

Viewed as classes, the following relationships hold with all the inclusions being proper (see [5; Proposition, p. 398]): inverse semigroups $\subset$ orthodox semigroups $\subset$ $\subset$ conventional semigroups $\subset$ regular semigroups. Natural questions arise as to what properties can be generalized from one class to another. This paper concentrates on describing certain homomorphisms. In particular, after the notation and preliminaries of section 2, the paper describes the following congruences (in order): the maximum idempotent-separating congruence on a conventional semigroup, the minimum orthodox congruence on a regular semigroup, and the minimum inverse and group congruence on a conventional semigroup. These congruences were chosen since most of the others are known; e.g., the minimum inverse and group congruences are described in [3], [6] and [7].

The paper's results are embodied in the four main theorems of section 3; remarks and corollaries that generalize results of [3], [6] and [7] appear after respective theorems.

## 2. NOTATION AND PRELIMINARY RESULTS

The notation of Clifford and Preston, ([1], [2]), will be used throughout the paper. If $S$ is regular and there is no danger of ambiguity, $E$ will be used to denote $E_S$, the set of idempotents of $S$. For each element $c$ of a regular semigroup $S$ define $V(c) =$ $= \{x \in S : cxc = c \text{ and } xcx = x\}$, the set of inverses of $c$.

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When a congruence $\varrho$ is such that $S/\varrho$ is the maximal homomorphic image of $S$ of type $C$, as in [2; p. 275] and [2; Theorem 11.25 (A), p. 276], then $\varrho$ will be called the *minimum congruence* on $S$ of type $C$ and $S/\varrho$ will be called the *maximum homomorphic image* of $S$ of type $C$. In other words, $S/\varrho$ is the maximum $C$-image if and only if $\varrho$ is of type $C$ and $\varrho \subseteq \sigma$ for each congruence $\sigma$ which is of type $C$.

Since this paper only treats a few specific congruences, the following symbols will be adopted:

$O_S$ = the identity congruence $\{(a, a) : a \in S\}$,
$\mu$ = the maximum idempotent-separating congruence,
$\lambda$ = the minimum orthodox semigroup congruence,
$\delta$ = the minimum inverse semigroup congruence,
$\sigma$ = the minimum group congruence.

If $\varrho$ is a relation on $S$ then $\varrho^*$ will denote the congruence on $S$ generated by $\varrho$.

The following results will also be needed.

**Lemma 2.1.** *Let $S$ be a regular semigroup and suppose that $xay = xby$ for some $a, b, x, y \in S$. Then there is a $u \in E$ such that $uau = ubu$.*

Proof. Multiplying by $x'$ and $y'$ we have that $x'xayy' = x'xbyy'$. By [4; Lemma 1.1, p. 146] there exists $u \in E$ such that $ux'x = yy'u = u$. Thus $uau = ubu$ and the lemma is established.

**Lemma 2.2.** *In a regular semigroup $S$, the following conditions are equivalent:*

(i) *$cEc' \subseteq E$ for all $c \in S$ and $c' \in V(c)$;*
(ii) *$eEe \subseteq E$ for all $e \in E$.*

Proof. It is clear that condition (i) implies condition (ii) since $e \in V(e)$. Conversely, let $c \in S$, $c' \in V(c)$, and $e \in E$. By condition (ii) $c'cec'c \in E$ and therefore

$$\begin{aligned}(cec')(cec') &= (cc'cec')(cec'cc'),\\ &= c(c'cec'cec'c)c',\\ &= c(c'cec'c)(c'cec'c)c',\\ &= c(c'cec'c)c' = cec'.\end{aligned}$$

Thus condition (ii) implies condition (i).

## 3. THE CONGRUENCES $\mu$, $\lambda$, $\delta$ AND $\sigma$

In this section results are presented which generalize some of those Meakin obtained in [6] and [7] for orthodox semigroups. The specific congruences treated are $\mu$, $\lambda$, $\delta$, and $\sigma$, in that order. First, though a lemma.

**Lemma 3.1.** *A homomorphic image of a conventional semigroup is conventional.*

Proof. Let $\phi$ be a homomorphism from the conventional semigroup $S$ onto the semigroup $S'$. Since $S$ is regular it follows that $S'$ is regular. Next, let $e'$ and $f'$ be arbitrary idempotents of $S'$. By Lallement's Lemma [3; Lemma 4.6, p. 52], $e'\phi^{-1}$ and $f'\phi^{-1}$ both contain idempotents of $S$, say $e \in e'\phi^{-1} \cap E$ and $f \in f'\phi^{-1} \cap E$. Since $S$ is conventional, $efe \in E$ and so $(efe)\,\phi = e'f'e' \in E'$. It follows by Lemma 2.2 that $S'$ is conventional and the proof is completed.

Recall next that a congruence on a semigroup $S$ is called *idempotent-separating* if each congruence class contains at most one idempotent of $S$. Lallement showed (see [3; Proposition 4.8, p. 52]) that any idempotent-separating congruence on a regular semigroup is contained in Green's equivalence $\mathscr{H}$. We make use of this result to determine the maximal idempotent-separating congruence on a conventional semigroup, thus generalizing the result of J. C. Meakin [6] from orthodox to conventional semigroups.

**Theorem 3.2.** *Let $S$ be a conventional semigroup and define the relation $\mu$ on $S$ by*

$$\mu = \{(a, b) \in S \times S\colon \text{there are inverses } a' \text{ of } a \text{ and } b' \text{ of } b \text{ such that } a'ea = b'eb \text{ and } aea' = beb' \text{ for all } e \in E\}\,.$$

*The relation $\mu$ is an idempotent-separating equivalence relation containing every idempotent-separating congruence on $S$. Thus $\mu$ generates the maximal idempotent-separating congruence on $S$.*

Proof. It is evident that $\mu$ is reflexive and symmetric. To show that $\mu$ is transitive note first that if $(a, b) \in \mu$, then $a'(aa'bb'aa')\,a = b'(aa'bb'aa')\,b$ since $aa'bb'aa' \in E$, where $a'$ and $b'$ are the inverses of $a$ and $b$ respectively which appear in the definition of $\mu$. Hence $a'bb'a = (b'aa'b)(b'aa'b) = b'aa'b$ since $b'aa'b \in E$. But $bb' \in E$, so $a'(bb')\,a = b'(bb')\,b = b'b$, and similarly $b'(aa')\,b = a'a$. Hence $a'a = b'b$. In a similar fashion, it is not difficult to see that $aa' = bb'$. From these two results we deduce that, in particular $\mu \subseteq \mathscr{H}$. We now proceed to the proof of the transitivity of $\mu$.

Suppose that $(a, b) \in \mu$ and $(b, c) \in \mu$. Then there are inverse $a'$ of $a$, $b'$ and $b^*$ of $b$, and $c^*$ of $c$ such that

$$a'ea = b'eb,\quad aea' = beb',\quad b^*eb = c^*ec\,,\quad beb^* = cec^*\,,\quad \text{for all}\quad e \in E\,.$$

In particular, we have seen that this implies that $aa' = bb'$, $a'a = b'b$, $cc^* = bb^*$, and $c^*c = b^*b$, and hence that $a$, $b$, and $c$ are $\mathscr{H}$-equivalent elements of $S$. Hence there are inverses $a^*$ of $a$ and $c'$ of $c$ such that

$$aa' = bb' = cc'\,,\quad a'a = b'b = c'c\,,$$

and

$$aa^* = bb^* = cc^*\,,\quad a^*a = b^*b = c^*c\,.$$

Now $a^*aa' \in V(a)$ and $c^*cc' \in V(c)$, and for all $e \in E$,

$$(a^*aa')\,ea = (a^*a)(a'ea) = (a^*a)(b'eb) = (b^*b)(b'ebb'b) =$$
$$= b^*(bb'ebb')\,b = c^*(bb'ebb')\,c = c^*(cc'ecc')\,c = (c^*cc')\,ec\,,$$

while

$$ae(a^*aa') = a(a^*aea^*a)\,a' = b(a^*aea^*a)\,b' =$$
$$= b(b^*beb^*b)\,b' = (beb^*)(bb') = (cec^*)(cc') = ce(c^*cc')\,.$$

Hence $(a, b) \in \mu$ and $\mu$ is transitive. That $\mu$ separates idempotents is obvious since we have already proved that $\mu \subseteq \mathscr{H}$.

Now let $\varrho$ be an idempotent-separating congruence of $S$. Then if $(a, b) \in \varrho$, we have that $(a, b) \in \mathscr{H}$, and hence there are inverses $a'$ of $a$ and $b'$ of $b$ such that $aa' = bb'$ and $a'a = b'b$. Then, since $(a, b) \in \varrho$, we have $(aa', ba') \in \varrho$, i.e. $(bb', ba') \in \varrho$, and hence $(b'bb', b'ba') \in \varrho$, i.e. $(b', a') \in \varrho$. Hence, for all $e \in E$ we have $(aea', beb') \in \varrho$, and so $aea' = beb'$ since both $aea'$ and $beb'$ are idempotents and $\varrho$ separates idempotents. Also $(b'eb, a'ea) \in \varrho$, and so $a'ea = b'eb$. It follows that $(a, b) \in \mu$, and consequently that $\varrho \subseteq \mu$. Thus $\mu$ is an idempotent-separating equivalence relation containing every idempotent-separating congruence on $S$.

Finally, denoting by $\bar{\mu}$ the congruence generated by $\mu$, we have $\mu \subseteq \bar{\mu}$. But by the preceding paragraph, $\bar{\mu} \subseteq \mu$. Thus $\mu$ is the maximal idempotent-separating congruence on $S$ and the proof is completed.

**Remark 3.3.** If $S$ is a conventional semigroup and $(x, y) \in \mu$, and if $x^*$ is an arbitrary inverse of $x$, then there exists an inverse $y^*$ of $y$ such that $xex^* = yey^*$ and $x^*ex = y^*ey$ for all $e \in E$.

Proof. If $(x, y) \in \mu$ and $x^*$ is an arbitrary inverse of $x$, then there are inverses $x'$ of $x$ and $y'$ of $y$ such that $xex' = yey'$ and $x'ex = y'ey$ for all $e \in E$. Also, since $(x, y) \in \mathscr{H}$, there is an inverse $y^*$ of $y$ such that $xx^* = yy^*$ and $x^*x = y^*y$. Thus for all $e \in E$,

$$xex^* = x(x^*xex^*xx'xx^*) = x(x^*xex^*x)\,x'(xx^*) =$$
$$= y(x^*xex^*x)\,y'(xx^*) = y(y^*yey^*y)\,y'(yy^*) = yey^*\,,$$

and dually, $x^*ex = y^*ey$.

When the preceding theorem is applied to orthodox semigroups, we have the following result.

**Corollary 3.4** [6; Theorem 4.4, p. 336] *The maximal idempotent-separating congruence on an orthodox semigroup is given by $\mu$.*

Proof. For $S$ orthodox, $x'y' \in V(yx)$ and it follows that $(c'a')\,e(ac) = (c'b')\,e(bc)$ and $(ac)\,e(c'a') = (bc)\,e(c'b')$ for all $e \in E$. Thus $\mu$ is right compatible. Dually $\mu$ is left compatible and is therefore a congruence.

It has been known for some time how to construct the minimum inverse congruence on an orthodox semigroup (see, for example, [3; Theorem 1.12, p. 190]). In generalizing this result, we seek to determine the minimum orthodox congruence on a conventional semigroup. A stronger result is presented from which the desired one will follow as a corollary.

**Theorem 3.5.** *On a regular semigroup $S$ define*

$$\lambda = \{(ef, efef) \in S \times S : e, f \in E\}, \tag{3.6}$$

*then $\lambda^*$, the congruence generated by $\lambda$, is the minimum orthodox congruence on $S$. Moreover, if $\varrho$ is any congruence on $S$ which contains $\lambda^*$, then $S/\varrho$ also is orthodox.*

Proof. It is clear that $\lambda^*$ is orthodox since $\lambda$ identifies $ef$ with $efef$. If $\varrho$ is any orthodox congruence on $S$, then $(ef, efef) \in \varrho$ implies that $\lambda \subseteq \varrho$. Thus $\lambda^* \subseteq \varrho$ and $\lambda^*$ is the minimum orthodox congruence on $S$.

Next, let $\varrho$ be a congruence containing $\lambda^*$ and let $e, f \in E$. Since $ef\lambda^*efef$, then $ef\varrho efef$; i.e., $S/\varrho$ is orthodox and the proof is completed.

Note in the preceding Theorem that if $S$ is an orthodox semigroup, then $ef = efef$ and therefore $\lambda = \lambda^* = O_S$; i.e., $\lambda$ reduces to the identity congruence on $S$.

**Corollary 3.7.** *If $S$ is a conventional semigroup, then the minimum orthodox congruence on $S$ is generated by the relation $\lambda$ of equation* (3.6).

Having described the minimum orthodox congruence on a conventional semigroup, the minimum inverse congruence is sought next.

**Theorem 3.8.** *On a regular semigroup $S$ define*

$$\delta = \{(a, b) \in S \times S : V(a) \cap V(b) \neq \emptyset\}.$$

*Then $\delta^*$, the congruence generated by $\delta$, is the minimum inverse semigroup congruence on $S$. Moreover, if $\varrho$ is any congruence on $S$ which contains $\delta^*$, then $S/\varrho$ also is inverse.*

Proof. If $a', a'' \in V(a)$, then $a \in V(a') \cap V(a'')$ and therefore $a'\delta a''$; i.e., $a'\delta^* a''$. Since $\delta^*$ identifies the inverses of $a$, $\delta^*$ is an inverse congruence on $S$.

Next let $\varrho$ be any inverse congruence on $S$ and suppose that $(a, b) \in \delta$. Then $V(a) \cap V(b) \neq \emptyset$ and therefore under $\varrho$, $V(a\varrho) = V(b\varrho)$. But then $a'\varrho \in V(b\varrho)$; i.e., $(a', b') \in \varrho$. Taking inverses, $(a, b) \in \varrho$ and thus we have that $\delta \subseteq \varrho$. It follows that $\delta^* \subseteq \varrho$ and therefore $S/\varrho$ is inverse.

Note that if $S$ is an inverse semigroup, $V(a) \cap V(b) \neq \emptyset$ implies that $V(a) = V(b)$. Thus $\delta = \delta^* = O_S$; i.e., $\delta$ reduces to the identity congruence on $S$. If $S$ is an orthodox semigroup, we have the following corollary.

**Corollary 3.9.** *If $S$ is an orthodox semigroup, then the minimum inverse congruence on $S$ is $\delta$.*

Proof. If $S$ is orthodox, then $V(a) \cap V(b) \neq \emptyset$ if and only if $V(a) = V(b)$. Thus $\delta$ is precisely the relation of [3; Theorem 1.12, p. 190] and the corollary follows.

The minimum group congruence on a conventional semigroup was described previously in [5] as

$$\{(a, b) \in S \times S : eae = ebe \text{ for some } e \in E\} . \tag{3.10}$$

Alternately, we have the following result which also generalizes the result of Meakin [7] from orthodox to conventional semigroups.

**Theorem 3.10.** *If $S$ is a conventional semigroup then the minimum group congurnece on $S$ is given by*

$$\sigma = \{(a, b) \in S \times S : V(ga) \cap V(gb) \neq \emptyset \text{ for some } g \in E\} .$$

Proof. Denote the congruence of (3.10) as $\beta$ and let $(a, b) \in \beta$. There then exists $e \in E$ such that $eae = ebe$. Denote $eae$ by $x$ and define $g = ex'e$ for any $x' \in V(x)$. We will show that $V(ga) \cap V(gb) \neq \emptyset$. First note that $exe = x$ and that $x^*xx^* = ex'e(eae)\,ex'e = ex'(eae)\,x'e = ex'xx'e = ex'e = x^*$. It follows then that $g^2 = xx^*xx^* = x(x^*xx^*) = xx^* = g$, and therefore

$$x^*(ga)\,x^* = x^*(xx^*a)\,x^* = x^*ax^* = ex'eaex'e = ex'xx'e = ex'e = x^* ,$$

while

$$ga(x^*)\,ga = xx^*ax^*xx^*a = xx^*ax^*a = x(x^*ax^*)\,a = xx^*a = ga .$$

Similarly, since $x = ebe$, $x^*(gb)\,x^* = x^*$ and $gb(x^*)\,gb = gb$. Thus $x^* \in V(ga) \cap V(gb)$ and $(a, b) \in \sigma$.

Conversely, suppose $(a, b) \in \sigma$; i.e., there exists an $f \in E$ such that $V(fa) \cap V(fb) \neq \emptyset$. For $x \in V(fa) \cap V(fb)$, define $g = fax$ and $h = xfa$ and note that $g^2 = faxfax = fax = g$ and $h^2 = xfaxfa = xfa = h$. We then have that

$$gf(a)\,h = faxf(a)\,xfa = fa(xfax)\,fa = fa(xfbx)\,fa = faxf(b)\,xfa = gf(b)\,h ,$$

and it follows by Lemma 2.1 that there exists $u \in E$ such that $uau = ubu$. Thus $(a, b) \in \beta$ and we conclude that $\sigma = \beta$.

Lastly, if the semigroup $S$ is orthodox, then in the preceding theorem, $V(fa) \cap V(fb) \neq \emptyset$ if and only if $V(fa) = V(fb)$ (see [3; Theorem 1.10, p. 189]). We thus have the following and concluding corollary.

**Corollary 3.11.** [7; Theorem 3.1, p. 264] *The minimum group congruence on an orthodox semigroup is*

$$\sigma = \{(a, b) \in S \times S : V(fa) = V(fb) \text{ for some } f \in E\} .$$

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