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HOMOPOLAR CIRCUITS IN POLAR GRAPHS

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1. INTRODUCTION

Polar graphs were introduced by F. Zítek in a lecture at the Conference on Graph Theory held at Štiřín in May 1972 (see also [9]). Most published work on the subject has been done by B. Zelinka [3–8], who also introduced the concept of homopolar paths [4]. In this paper we consider mainly homopolar circuits.

A polar graph may be thought of as being a graph in which each vertex has two poles such that an edge incident with a vertex is incident with just one of its poles. A formal definition will be given below. We shall be concerned with circuits in which the two edges incident with a vertex are always incident with the same pole of the vertex. Since a loop joining one pole of a vertex to the other could never be part of such a circuit, and since a loop joining a pole to itself together with the vertex would form such a circuit, loops are not interesting and we shall consider polar graphs without loops. Also, two distinct edges joining the same pole of one vertex to the same pole of another would, together with the two vertices, form a homopolar circuit. Therefore we shall not allow this kind of multiple edge.

We can now state our definition as follows (Zelinka [3]). A *polar graph* G is an ordered quintuple $G = (V, E, P, p, q)$, where V, E, P are mutually disjoint finite sets, p is a mapping from V into the set of unordered pairs of distinct elements of P and q is a mapping from E into the set of unordered pairs of distinct elements of P such that (1)–(4) are satisfied:

(1) for any $u, v \in V$, $u \neq v$: $p(u) \cap p(v) = \emptyset$;

(2) for any $e \in E$ and any $v \in V$: $p(v) \neq q(e)$;

(3) for any $e, f \in E$, $e \neq f$: $q(e) \neq q(f)$;

(4) for any $a \in P$ there is a $v \in V$ such that $a \in p(v)$.

The elements of $V = V(G)$ are called *vertices*, the elements of $E = E(G)$ *edges* and the elements of $P = P(G)$ *poles*. For $v \in V$ the elements of $p(v)$ are called the *poles of* v. We say that an edge e is *incident* with the poles of $q(e)$, and we also say that e *joins* the two poles of $q(e)$. If u and v are vertices such that e joins a pole of u and a pole of v, then we shall also say that e joins u and v and that e is incident with u and v. The *degree* $d(v)$ *of a vertex* v is the number of edges incident with it, and the *degree* $d_p(a)$ *of a pole* a is the number of edges incident with a. Thus if $p(v) = \{a, b\}$ we have that $d(v) = d_p(a) + d_p(b)$. We shall illustrate a vertex as a rhombus partitioned into two triangles, the poles of the vertex (see e.g. Figure 1). The *order* of a polar graph G is $|V(G)|$, and the minimum degree of a vertex in G will be denoted by $\delta(G)$.

A *homopolar path* joining poles a and b is a sequence $v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k$, $k \geqq 2$, where $v_1, \ldots, v_k$ are distinct vertices and $e_1, \ldots, e_{k-1}$ are edges of G, e_1 is incident with the pole a of v_1, e_{k-1} is incident with the pole b of v_k, and for $i = 2, \ldots$ $\ldots, k-1$, e_{i-1} and e_i are incident with the same pole of v_i. A *homopolar circuit* is defined in the same way, except that $v_1 = v_k$ and $a = b$. It is a *homopolar hamilton circuit* if $V(G) = \{v_1, \ldots, v_{k-1}\}$.

Let $t(n)$ be the greatest number of edges in a polar graph of order n which admits no homopolar circuits. The main object of this paper is to show that $t(n) = 4n - 4$ and to characterize those polar graphs of order n with $t(n)$ edges which admit no homopolar circuits. The quantity $t(n)$ was also found by Roland Häggkvist (private communication).

We also determine δ^*, the greatest value of the least vertex degree of a polar graph with no homopolar circuits. Finally we give a sufficient condition for a polar graph to admit a homopolar hamilton circuit.

Before we prove these results we need some more terminology. If G is a polar graph, then the *underlying graph* $\mathbf{G}$ is the (ordinary simple) graph with vertex set $V(G)$ in which two vertices are joined by a single edge if and only if they are joined in G. A polar subgraph of a polar graph is is defined in the obvious way. We shall not distinguish between a sequence forming a homopolar path or circuit and the corresponding polar subgraph.

In the following we shall write HPP, HPC and HPHC for homopolar path, homopolar circuit and homopolar hamilton circuit, respectively.

2. RESULTS ABOUT THE EXISTENCE OF AN HPC

A *polarization* of a polar graph $G = (V, E, P, p, q)$ is an ordering of each set $p(v)$, $v \in V$. Given a polarization we shall call the first element of $p(v)$ the *N-pole* of v, and denote it by v_N, and the second the *S-pole* of v, denoted by v_S. For each pole a, we let $d_N(a)$ and $d_S(a)$ be the number of edges joining a to an N-pole and to an S-pole respectively.

Lemma 1. *A polar graph G has a polarization satisfying*

$$d_N(v_N) + d_S(v_S) \geqq \tfrac{1}{2}\, d(v) \quad \text{for all} \quad v \in V(G)\,. \tag{5}$$

Further, G contains a polar subgraph H with the property that at least one pole of each vertex has degree zero in H, and for which

$$|E(H)| \geqq \tfrac{1}{4}|E(G)| . \tag{6}$$

H can be chosen such that there is strict inequality in (6), *unless each pair of vertices that are joined by an edge in G are joined by either two or four edges as in Figure* 1.

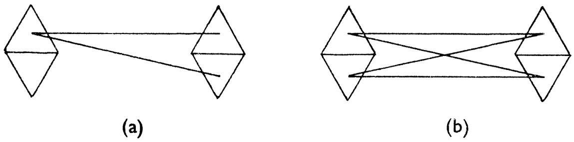

Fig. 1.

Proof. Consider a polarization of G such that the number x of edges joining two N-poles or two S-poles is maximum. We claim that it satisfies (5). Suppose that for some vertex v, (5) fails. Then

$$\begin{aligned} d_S(v_N) + d_N(v_S) &= d_p(v_N) - d_N(v_N) + d_p(v_S) - d_S(v_S) = \\ &= d(v) - (d_N(v_N) + d_S(v_S)) > \tfrac{1}{2}\, d(v) . \end{aligned}$$

Thus reversing the ordering of $p(v)$ would create a polarization with more edges joining two N-poles or two S-poles, which contradicts the maximality of x.

Obviously $x \geqq \frac{1}{2}|E(G)|$, and so either the number x_N of edges joining two N-poles or the number x_S of edges joining two S-poles is at least $\frac{1}{4}|E(G)|$. If $x_N > \frac{1}{4}|E(G)|$ we let H be the subgraph containing only edges joining two N-poles, if not we let H contain only edges joining two S-poles. Clearly H satisfies (6). Suppose that there is equality. Then there is equality in (5) for all $v \in V(G)$. Note that we can reverse the ordering of $p(v)$ for any vertex v, redefine H as above and still have the same x and therefore (5) and (6) fulfilled. Assume that two vertices u and w are joined by at least one edge in G, but not as in Figure 1. Then, by reversing orderings at either u or w if necessary, we may assume that more edges join u_N to w_N or u_S to w_S, than u_N to w_S or u_S to w_N. Since there is equality in (5) for u, u is joined to some other vertex v in such a way that more edges join u_N to v_S or u_S to v_N, than u_N to v_N or u_S to v_S. We reverse the ordering of $p(v)$. (5) is still satisfied; but now with strict inequality for u. As remarked above, this contradicts that $x = \frac{1}{2}|E(G)|$. This proves the lemma.

Theorem 2. $t(n) = 4n - 4$.

Proof. (i) Let T be any tree of order n, and let G be a maximal polar graph whose underlying graph is isomorphic to T, i.e. G is obtained from T by replacing each edge by four edges as in (b) of Figure 1. Then G has $4n - 4$ edges and contains no HPC. Hence $t(n) \geqq 4n - 4$.

(ii) Let G be a polar graph with n vertices and at least $4n - 3$ edges. Let H be a polar subgraph of G as in Lemma 1. By (6), H has at least n edges. Hence H is a graph of order n with at least n edges, and so H contains a circuit. This clearly corresponds to an HPC in G, and so $t(n) \leqq 4n - 4$.

This proves the theorem.

We shall now proceed to find all polar graphs of order n which have $4n - 4$ edges and admit no HPC. Let the class of all such polar graphs be $\mathscr{E}_n$, and let $\mathscr{E} = \bigcup_{n \geqq 1} \mathscr{E}_n$. Part (i) of the proof of Theorem 2 shows that any maximal polar graph whose underlying graph is a tree is in $\mathscr{E}$. There are other polar graphs in $\mathscr{E}$, however.

Let $\mathscr{F}$ be the minimal class of polar graphs satisfying (7)–(9):

(7) $\mathscr{F}$ contains the polar graph with just one vertex;

(8) if $G \in \mathscr{F}$ and $u, v \in V(G)$ (we allow $u = v$), then any polar graph obtained from G by adding a vertex $w \notin V(G)$, joining one pole of w to both poles of u and joining the other pole of w to both poles of v, is in $\mathscr{F}$;

(9) if $G_1, G_2 \in \mathscr{F}$ and $|V(G_1) \cap V(G_2)| = 1$, then $G_1 \cup G_2 \in \mathscr{F}$.

Let $\mathscr{F}_n$ be the class of all polar graphs of order n which are in $\mathscr{F}$. Note that since $u = v$ is allowed in (8), $\mathscr{F}$ contains each maximal polar graph whose underlying graph is a tree. It is not hard to see that $\mathscr{F} \subseteq \mathscr{E}$. We shall show that $\mathscr{E} = \mathscr{F}$.

Lemma 3. *If $G \in \mathscr{F}$ and $|V(G)| \geqq 2$, then any pair of vertices are joined by an even number of edges, and so each vertex has even degree. Furthermore, either $\delta(G) = 4$, or $\delta(G) \geqq 6$ and G has at least one cut-vertex.*

Proof. Obvious from the definition of $\mathscr{F}$.

Lemma 4. *If $G \in \mathscr{E}$ and a and b are two poles of distinct vertices of G, then G admits an* HPP *joining a and b.*

Proof. If there is an edge joining a and b in G, then there is an HPP joining a and b with just this one edge. If there is no such edge, then it follows from Theorem 2 that the polar graph obtained from G by adding an edge joining a and b admits an HPC, which clearly includes this edge. The rest of the circuit forms an HPP in G joining a and b.

Theorem 5. $\mathscr{E} = \mathscr{F}$.

Proof. We shall show that $\mathscr{E}_n = \mathscr{F}_n$ for all $n \geqq 1$ by induction on n. It is clear that $\mathscr{E}_1 = \mathscr{F}_1$. Suppose that $n > 1$ and that $\mathscr{E}_{n'} = \mathscr{F}_{n'}$ for all $n' < n$. We already know that $\mathscr{F}_n \subseteq \mathscr{E}_n$ and so it is sufficient to show that $\mathscr{E}_n \subseteq \mathscr{F}_n$.

Let $G \in \mathscr{E}_n$. We must show that $G \in \mathscr{F}_n$. Let H be as in Lemma 1. We cannot have strict inequality in (6), because then we would have $|E(H)| \geqq n$ and so H would contain a circuit, which would correspond to an HPC in G. It follows that if a pair of vertices are joined by an edge in G, they are joined as in (a) or (b) of Figure 1. So each vertex of G has even degree. Lemma 4 implies that G is connected. Since $|E(G)| = 4n - 4$, it follows that $\delta(G) < 8$, and so $\delta(G)$ is 2, 4 or 6. If $\delta(G) = 2$, however, it follows immediately by deleting a vertex of degree 2 from G and using Theorem 2 that G has an HPC, which is a contradiction. Hence $\delta(G) = 4$ or $\delta(G) = 6$. We shall consider the following mutually exclusive cases:

CASE A: G has a cut-vertex.

CASE B (i), $i = 4, 6$: G has no cut-vertices and $\delta(G) = i$.

CASE A: G has a cut-vertex.

In this case G can be expressed as the union of two polar graphs G_1 and G_2 such that $|V(G_1)| < n$, $|V(G_2)| < n$ and $|V(G_1) \cap V(G_2)| = 1$. Because G has no HPC, G_1 and G_2 have no HPC, and so, by Theorem 2, $|E(G_1)| \leqq 4|V(G_1)| - 4$ and $|E(G_2)| \leqq 4|V(G_2)| - 4$. Hence

$$4n - 4 = |E(G)| = |E(G_1)| + |E(G_2)| \leqq 4|V(G_1)| - 4 + 4|V(G_2)| - 4 =$$
$$= 4(|V(G_1)| + |V(G_2)| - 1) - 4 = 4|V(G)| - 4 = 4n - 4\,,$$

and so we must have

$$|E(G_1)| = 4|V(G_1)| - 4$$

and

$$|E(G_2)| = 4|V(G_2)| - 4\,.$$

By the inductive hypothesis $G_1 \in \mathscr{F}$ and $G_2 \in \mathscr{F}$ and so, by (9), $G \in \mathscr{F}$.

CASE B(4): G has no cut-vertices and $\delta(G) = 4$.

Let w be a vertex of degree 4 and let G_w be the polar graph obtained from G by deleting w. G_w admits no HPC and has $4(n - 1) - 4$ edges, and so $G_w \in \mathscr{E}_{n-1}$. By the inductive hypothesis, $G_w \in \mathscr{F}_{n-1}$. If w is joined to vertices u and v as in (8), then $G \in \mathscr{F}_n$. Suppose that this is not the case. Then there is a pole of w joined to poles a and b of distinct vertices. By Lemma 4, G_w admits an HPP joining a and b. Clearly this implies that G admits an HPC, which is a contradiction.

CASE B(6): G has no cut-vertices and $\delta(G) = 6$.

We wish to show that this case can not occur. We consider two subcases which are exhaustive (since each pair of vertices are joined by an even number of edges):

a) there is a vertex of degree 6 joined to only two vertices;
b) all vertices of degree 6 are joined to exactly three vertices.

Subcase a): Let w be a vertex of degree 6 joined to only two vertices. Let u be the vertex joined to w by four edges, v the vertex joined to w by two edges. We have one of the situations of Figure 2. The figure also illustrates part of H, the polar graph obtained from G by deleting w and adding two edges joining u and v such that the two edges are joined to different poles of u and such that the poles of v have the same degree in H as in G. Note that none of these edges could be in G, since G has no HPC.

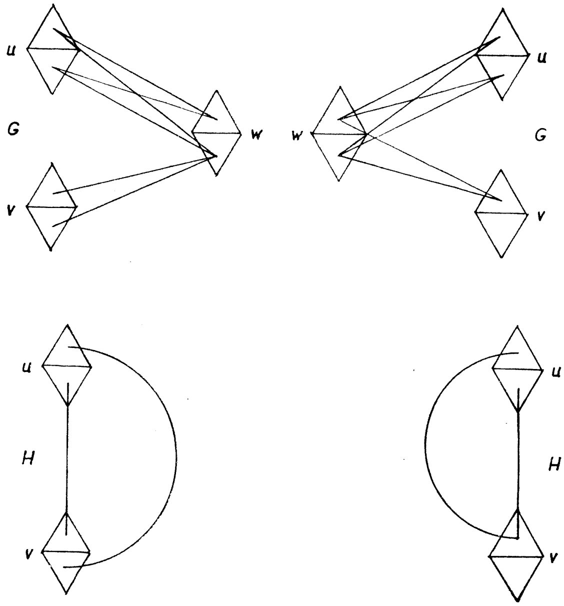

Fig. 2.

It is easy to see that H admits no HPC, because an HPC would include exactly one of the new edges and would immediately imply an HPC in G, which is impossible. Since $|E(H)| = |E(G)| - 4$, it follows that $H \in \mathscr{E}_{n-1}$. By the inductive hypothesis $H \in \mathscr{F}_{n-1}$.

H has no cut-vertex, because a cut-vertex of ***H*** would be a cut-vertex of ***G***. By Lemma 3, H contains a vertex of degree 4. Since $\delta(G) = 6$, this vertex must be ***u***. It follows that u has degree 6 in G, and so u is joined to just one vertex z different from w. We have $z \neq v$, for otherwise v would be a cut-vertex of ***G***. Let H' be obtained from H by deleting u. Then $|V(H')| = n - 2$ and $|E(H')| = |E(G)| - 8 = 4(n-2) - - 4$, and H' is a polar subgraph of G. Since H' admits no HPC, it belongs to $\mathscr{E}_{n-2}$. By Lemma 4, H' admits an HPP joining any pole of v to any pole of z. Taking these poles to be poles joined to w and u respectively, one obtains an HPC in G, which is a contradiction.

Subcase b): Assume that all vertices of degree 6 have exactly three neighbours. We consider three sub-subcases:

(i) there is a vertex of degree 6 with a pole joined to poles of three distinct vertices;
(ii) there is a vertex of degree 6 with a pole joined to both poles of each of two distinct vertices;
(iii) there are no vertices of degree 6 satisfying (i) or (ii).

Sub-subcase (i): Let v be a vertex of degree 6 in G and let a be a pole of v joined to poles b_1, b_2, b_3 of three distinct vertices. Since G does not have an HPC, it has no edge joining b_1 and b_2 and no edge joining b_2 and b_3. Let H be the polar graph obtained from G by deleting v and adding an edge joining b_1 and b_2 and an edge joining b_2 and b_3. Then $|V(H)| = n - 1$ and $|E(H)| = 4n - 4 - 6 + 2 = t(n - 1)$, and so either H has an HPC or $H \in \mathscr{E}_{n-1}$. In the former case the edges of the HPC which are in G and two of the edges joining a to b_1, b_2, and b_3 are the edges of an HPC in G, which is impossible. And the latter case is excluded by the fact that in H the vertex to which b_1 belongs has odd degree, contradicting the inductive hypothesis together with Lemma 3.

Sub-subcase (ii): Let v be a vertex of degree 6 and let a be a pole of v which is joined to both poles of each of the vertices u and w. Since G admits no HPC it contains no edge joining u and w. Considering the polar graph obtained by adding 4 such edges (3 will suffice) and deleting v one immediately obtains a contradiction.

Sub-subcase (iii): Let v be any vertex of degree 6 in G. Since sub-subcases (i) and (ii) do not apply, it follows that we have the situation of Figure 3. Let the vertices u_1, u_2, u_3, and the poles a, b, a_1, b_1, a_2, b_2, a_3, b_3 be as indicated in the figure.

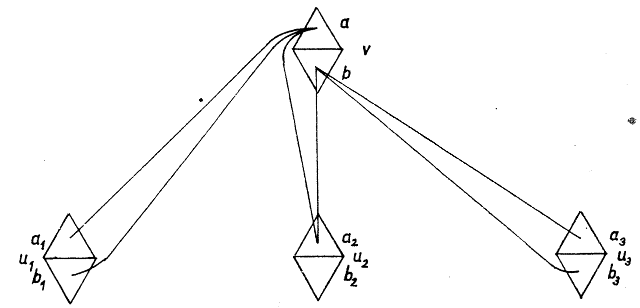

Fig. 3.

Let H be obtained from G by deleting v. Since G has no cut-vertices, H is connected and so H contains either a path joining u_1 and u_2 which does not contain u_3, or a path joining u_2 and u_3 which does not contain u_1. Suppose that the latter is the case.

Since G has no HPC, it contains no edge joining a_2 to a pole of u_1 or u_3. For $j = 1, 3$, let H_j be the polar graph obtained from H by adding edges joining a_2 to both poles of u_j. It is easy to see that $H_1, H_3 \in \mathscr{E}_{n-1}$. By the induction hypothesis $H_1, H_3 \in \mathscr{F}_{n-1}$.

Suppose that $\boldsymbol{H}_1$ has a cut-vertex w, say. Then H_1 can be expressed as the union of polar graphs H' and H'' such that $|V(H')| < n - 1$, $|V(H'')| < n - 1$ and $V(H') \cap \cap V(H'') = \{w\}$, and the notation may be chosen such that u_1 and u_2 lie in H'. Since $H_1 \in \mathscr{F}_{n-1}$ it follows that H' and H'' belong to $\mathscr{F}$, and so $H', H'' \in \mathscr{E}$. If $u_3 \in V(H')$, then w is a cut-vertex of $\boldsymbol{G}$, which is contrary to the hypothesis that $\boldsymbol{G}$ has no cut-vertices. Hence $u_3 \in V(H'') \setminus \{w\}$. In particular we note that $w \neq u_3$. Now $H_3 \in \mathscr{F}_{n-1}$, H_3 contains $H'' \in \mathscr{E}$ and H_3 contains the edges joining a_2 to a_3 and b_3; these edges are not in H''. By Lemma 4, u_2 and u_3 cannot both be in H'', since this would imply the existence of an HPP in H'' which could be extended to an HPC in H_3. Hence $u_2 \notin V(H'')$, and so $w \neq u_2$. Since w separates u_2 and u_3 in $\boldsymbol{H}_1$, and so in $\boldsymbol{H}$, and since $\boldsymbol{H}$ contains a path joining u_2 and u_3, which does not contain u_1 we deduce that $u_1 \neq w$. It follows that the vertices u_1, u_2, u_3, and w are distinct. We illustrate the situation in Figure 4.

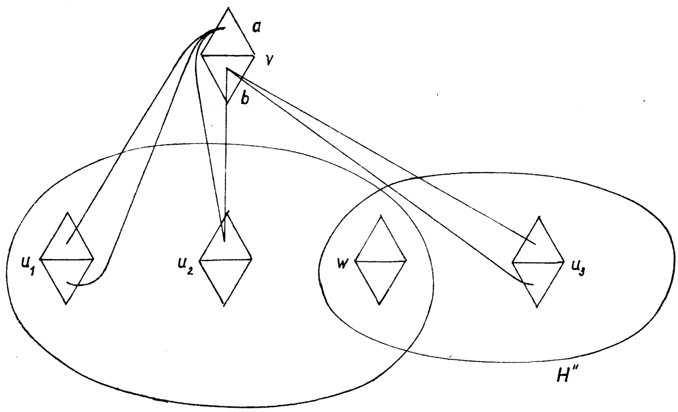

Fig. 4.

Let G' be obtained from G by deleting all vertices in $V(H'') \setminus \{w\}$ and by adding two edges joining b to each pole of w. Suppose that $|V(H')| = n'$. Then $|V(G')| = = n' + 1$. Since $H' \in \mathscr{E}_{n'}$ it follows that $|E(G')| = 4n' - 4 - 2 + 6 = t(n' + 1)$. Hence either G' has an HPC or $G' \in \mathscr{E}_{n'+1}$. Suppose first that G' has an HPC. Since G has no HPC, such an HPC must include an edge joining b to a pole of w, say the pole b'. Since $H'' \in \mathscr{E}$, there is by Lemma 4 an HPP in H'' joining b' and a pole of u_3. This implies the existence of an HPC in G, which is impossible. Hence G' does not have an HPC, and so $G' \in \mathscr{E}_{n'+1}$. By the inductive hypothesis $G' \in \mathscr{F}_{n'+1}$. Clearly each vertex of G' different from w has degree at least 6 in G'. Since $w \in H'$ and $H' \in \mathscr{F}$, by Lemma 3 $\delta(H') \geqq 4$, and so w has degree at least 6 in G'. Hence $\delta(G') \geqq 6$.

It follows from Lemma 3 that G' has a cut-vertex $w' \neq v$. Now it is easy to see that w' would also be a cut-vertex of G, which contradicts the hypothesis that G has no cut-vertices.

This contradiction enables us to deduce that H_1 does not have any cut-vertices. Since $H_1 \in \mathscr{F}$ it follows from Lemma 3 that H_1 has a vertex of degree 4. This vertex can only be u_3; it follows that u_3 has degree 6 in G.

Hence from the assumption that H has a path joining u_2 and u_3 not containing u_1 we have deduced that u_3 has degree 6. Similarly from the alternative possibility that H has a path joining u_1 and u_2 not containing u_3 we can deduce that u_1 has degree 6 in G. It follows, therefore, that at least one of u_1 and u_3 has degree 6 in G. Since v was chosen to be an arbitrary vertex of degree 6 in G we deduce that if u is any vertex of degree 6 in G, then some pole of u is joined to both poles of another vertex of degree 6. For each u, let $f(u)$ be one such vertex. Then let v_1 be some vertex of degree 6, and let $v_i = f(v_{i-1})$, $i = 2, 3, \ldots$. Since $|V(G)|$ is finite, there exist integers $j_1 < j_2$ such that $v_{j_1} = v_{j_2}$; we may suppose that j_1 and j_2 are chosen to make $j_2 - j_1$ as small as possible. It is easy to see that G contains an HPC with vertices $v_{j_1}, \ldots, v_{j_2-1}$. This is a contradiction.

This final contradiction completes the discussion of sub-subcase (iii), and so of subcase b) of CASE B(6). Having discussed all possible cases we conclude that $G \in \mathscr{F}_n$. The truth of the theorem now follows by induction.

We complete our discussion of the existence of HPC's in polar graphs by proving that $\delta^* = 6$.

Theorem 6. $\delta^* = 6$.

Proof. (i) That $\delta^* \geqq 6$ follows from the fact that $\mathscr{F}$ contains polar graphs whose minimum degree is 6, for example that shown in Figure 5.

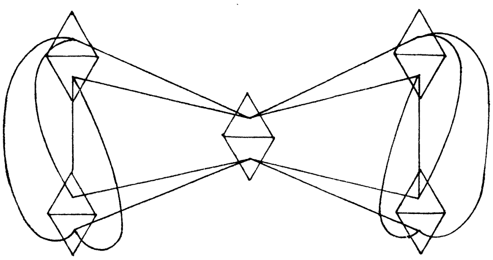

Fig. 5.

(ii) Let G be a polar graph such that $\delta(G) \geqq 7$. Give G a polarization satisfying (5) of Lemma 1. Let G' be the polar subgraph containing all edges of G joining two N-poles or two S-poles. By (5), $\delta(G') \geqq 4$, and so $|E(G')| \geqq 2|V(G)|$. It follows that either at least $|V(G)|$ edges join two N-poles or at least $|V(G)|$ edges join two S-poles. The existence of an HPC in G is easily deduced. Hence $\delta^* \leqq 6$.

3. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF AN HPHC

We give two conditions, each of which is sufficient to ensure the existence of an HPHC in a polar graph. The first condition, (10) below, was first proved by the second author in [2], where the relationship between 2-round graphs and HPHC's in polar graphs is discussed. We present a proof of (10) in this paper, partly because it is short, and partly because the result is not stated in this way in [2], where it occurs in a different setting.

Theorem 7. *A polar graph of order n has an* HPHC *if either of the following conditions is satisfied*:

(10) *each vertex v has degree $d(v) \geqq \frac{1}{3}(8n - 7)$*;

(11) *for each vertex v and each pole $a \notin p(v)$ such that a is not joined by an edge to any pole of v, there is a pole $b \in p(v)$ such that $d_p(a) + d_p(b) \geqq 3n - 2$.*

Proof. (10): Assume that G is a polar graph of order n in which each vertex has degree at least $\frac{1}{3}(8n - 7)$. By (5) of Lemma 1, G has a polarization such that

$$d_N(v_N) + d_S(v_S) \geqq \frac{8n - 6}{6} = \frac{4n}{3} - 1 \quad \text{for all} \quad v \in V(G)\,. \tag{12}$$

Let G_N be the polar subgraph of G containing exactly all edges joining two N-poles, and let G_S be the polar subgraph of G containing exactly all edges joining two S-poles. Since

$$d_N(v_N) \geqq \frac{4n}{3} - 1 - d_S(v_S) \geqq \frac{4n}{3} - 1 - (n - 1) = \frac{n}{3}, \quad \text{for all} \quad v \in V(G),$$

we have that $\delta(G_N) \geqq \frac{1}{3}n$. Similarly $\delta(G_S) \geqq \frac{1}{3}n$.

It follows from (12) that for each vertex v either $d_N(v_N) \geqq \frac{1}{6}(4n - 2) = \frac{1}{3}(2n - 1)$ or $d_S(v_S) \geqq \frac{1}{3}(2n - 1)$. Hence either G_N or G_S contains at least $\frac{1}{2}n$ vertices of degree at least $\frac{1}{3}(2n - 1)$. By a theorem of Chvátal (Theorem 1 in [1]), either G_N or G_S contains a hamilton circuit. This obviously corresponds to an HPHC in G, which proves (10).

(11): Assume that G' satisfies (11) and that G' does not contain an HPHC. By adding edges to G' we can obtain an edge-maximal polar graph G which satisfies (11) and contains no HPHC. It is easy to see that since G does not contain an HPHC, it must have a vertex v and a pole $a \notin p(v)$ such that a is not joined by an edge to any pole of v. By (11), there is a pole $b \in p(v)$ such that $d_p(a) + d_p(b) \geqq 3n - 2$.

By the maximal property of G, there is homopolar path joining a and b and containing all vertices of G. Let the vertices of this path be, in order, $v_1, \ldots, v_n = v$ such that $a \in p(v_1)$ and $b \in p(v_n)$. Give G a polarization in which the N-pole of each vertex is the pole incident with the edges of the path. Then, for each $i = 2, \ldots, n - 1$, if a is joined to the N-pole of v_i, then b is not joined to the N-pole of v_{i-1}, because otherwise G would contain an HPHC with vertices $v_1, \ldots, v_{i-1}, v_n, v_{n-1}, \ldots, v_i, v_1$. It follows that

$$d_N(b) \leqq n - 1 - d_N(a),$$

and so

$$d_p(a) + d_p(b) = d_N(a) + d_N(b) + d_S(a) + d_S(b) \leqq$$
$$\leqq (n - 1) + (n - 1) + (n - 1) = 3n - 3$$

contradictory to hypothesis. This proves (11).

Corollary 8. *If G is a polar graph of order n in which each pole a has degree $d_p(a) \geqq \frac{1}{6}(8n - 7)$, then G contains an* HPHC.

The bound on $d_p(a)$ in Corollary 8 is approximately $\frac{4}{3}n$. In [2] it is conjectured that $d_p(a) \geqq n$ for each pole would be a sufficient condition to ensure the existence of an HPHC.

4. A FINAL REMARK

Zelinka [3–4] has also been interested in *polarized* graphs, i.e. polar graphs with a polarization. We note that our theorems all hold for polarized graphs as well.

References

[1] *V. Chvátal:* On Hamilton's ideals, J. Combinatorial Theory B *12* (1972), 163—168.

[2] *D. D. Grant:* On certain cycles in graphs, Submitted.

[3] *B. Zelinka:* Isomorphisms of polar and polarized graphs, Czech. Math. J. *26* (*101*) (1976), 339—351.

[4] *B. Zelinka:* Analoga of Menger's theorem for polar and polarized graphs, Czech. Math. J. *26* (*101*) (1976), 352—360.

[5] *B. Zelinka:* Eulerian polar graphs, Czech. Math. J. *26* (*101*) (1976), 361—364.

[6] *B. Zelinka:* Self-derived polar graphs, Czech. Math. J. *26* (*101*) (1976), 365—370.

[7] *B. Zelinka:* Groups and polar graphs, Čas. Pěst. Mat. *101* (1976), 2—6.

[8] *B. Zelinka:* Polar graphs and railway traffic, Apl. Mat. *19* (1974), 169—176.

[9] *F. Zítek:* Some Remarks on Polar Graphs, Recent Advances in Graph Theory (Proc. Second Czechoslovak Symp., Prague, 1974), 537—539. Academia, Prague, 1975.

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