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ON VALUE SELECTORS AND TORSION CLASSES OF LATTICE ORDERED GROUPS

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In this note we will investigate a problem proposed by J. Martinez [7] on the relation between value selectors and torsion classes of lattice ordered groups.

1. PRELIMINARIES

We shall use the standard notation for lattice ordered groups (cf. Conrad [1] and Fuchs [2]). The group operation will be written additively.

The system of all convex $l$-subgroups of a lattice ordered group $G$ will be denoted by $c(G)$; this system is partially ordered by inclusion. Then $c(G)$ is a complete lattice; the lattice operations in $c(G)$ are denoted by $\wedge$, $\vee$.

In what follows we shall consider objects belonging to some type of the following hierarchy:

1) lattice ordered groups and their elements;
2) classes of lattice ordered groups;
3) classes of classes of lattice ordered groups.

Let $\mathcal{G}$ be the class of all lattice ordered groups. Let $A$ be a nonempty subclass of $\mathcal{G}$. Consider the following conditions for $A$:

(a) If $G \in \mathcal{G}$ and if $\{H_i\}_{i \in I} \subseteq A \cap c(G)$, then $\bigvee_{i \in I} H_i \in A$.
(b) If $G \in A$ and $H \in c(G)$, then $H \in A$.
(c) $A$ is closed with respect to homomorphisms.

The class $A$ is said to be a torsion class, if it satisfies (a), (b) and (c) (cf. Martinez [5], [6], [7]; a different terminology (using the term 'hereditary torsion class') has been applied in [4], [8]). Each variety of lattice ordered groups is a torsion class (Holland [3]).

Let $T$ be the class of all torsion classes; $T$ is partially ordered by inclusion. Then $T$ is a complete lattice [5]. Several properties of the lattice $T$ were established in [5], [9].
2. VALUE SELECTORS

The notion of a values selector was introduced in [7]. Let us recall some definitions and results concerning this notion.

Let \( G \in \mathcal{G} \), \( x \in G \). A convex \( l \)-subgroup of \( G \) maximal with respect to the property of noncontaining \( x \) is called a value of \( x \). A convex \( l \)-subgroup of \( G \) is said to be a value if it is a value of an element of \( G \). Let \( M_0(G) \) be the set of all values of \( G \).

A value selector is a function \( M \) assigning to each lattice ordered group \( G \) a subset \( M(G) \) of \( M_0(G) \) such that the following conditions are fulfilled:

1. If \( H \in c(G) \), then \( M(H) = \{ C \cap H : C \in M(G) \) and \( C \nsubseteq H \} \).
2. If \( K \) is an \( l \)-ideal of \( G \), then \( M(G/K) \supseteq \{ C/K : C \in M(G) \) and \( C \supseteq K \} \).

(Of course, we also assume that the mapping \( M \) is defined intrinsically, i.e., if \( \varphi \) is an isomorphism of a lattice ordered group \( G_1 \) onto a lattice ordered group \( G_2 \), then \( M(G_2) = \{ \varphi(C) : C \in M(G_1) \} \).

Let \( M_1 \) and \( M_2 \) be value selectors. We put \( M_1 \leq M_2 \) if \( M_1(G) \subseteq M_2(G) \) for each lattice ordered group \( G \). Let \( \{ M_i \}_{i \in I} \) be a family of value selectors; we define \( M_i(G) = \cap_{i \in I} M_i(G) \) and \( M_i(G) = \cup_{i \in I} M_i(G) \) for each \( G \in \mathcal{G} \). Then \( M_1 \) and \( M_2 \) are value selectors, and \( M_1 = \bigwedge_{i \in I} M_i \), \( M_2 = \bigvee_{i \in I} M_i \).

Let \( M \) be a value selector. We denote by \( T(M) \) the class of all lattice ordered groups \( G \) such that \( M(G) = M_0(G) \). For each torsion class \( A \) and each \( G \in \mathcal{G} \) we put

\[
A^\wedge(G) = \{ H \in M_0(G) : A(G) \nsubseteq H \},
\]

where \( A(G) \) is the join of all convex \( l \)-subgroups of \( G \) belonging to \( A \).

Then we have (cf. [7]; Lemmas 1.1—1.3):

**2.1. Lemma.** For each value selector \( M \), \( T(M) \) is a torsion class.

**2.2. Lemma.** For each torsion class \( A \), \( A^\wedge \) is a value selector; moreover, for \( G \in \mathcal{G} \) we have \( G \in A \) if and only if \( A^\wedge(G) = M_0(G) \).

**2.3. Lemma.** If \( A \) is a torsion class and \( M \) is a value selector, then \( T(M)^\wedge \leq M \) and \( T(A^\wedge) = A \).

The following problem has been proposed in [7]:

'The function \( M \to T(M) \) preserves arbitrary intersections. But it is unknown whether it also preserves joins. It would be of interest to know it, for it would shed light on the following question: If \( A \) is a torsion class, is there a largest value selector \( M \) such that \( T(M) = A^\wedge \)? There is always a smallest, namely \( A^\wedge \). In view of the inequality in 1.3, the author doubts that it preserve joins.'
3. THE MAPPINGS \( s_1 \) AND \( s_2 \)

Let \( G \in \mathcal{G} \) and \( X \subseteq G \). We denote \( X^\delta = \{ g \in G : |g| \land |x| = 0 \text{ for each } x \in X \} \). If we consider several lattice ordered groups then we sometimes write \( X^\delta(G) \) rather than \( X^\delta \). It is well-known that \( X^\delta \) is a convex \( l \)-subgroup of \( G \).

The following lemma is easy to verify.

3.1. Lemma. Let \( 0 < x \in G \) and suppose that the interval \([0, x]\) is a chain. Then \( \{x\}^\delta \) is a linearly ordered group.

3.2. Lemma. Let \( 0 < x \in G \) and suppose that the interval \([0, x]\) is a chain. Then \( x \) possesses a unique value \( B + \{x\}^\delta \), where \( B \) is the value of \( x \) in \( \{x\}^\delta \).

Proof. Put \( \{x\}^\delta = A \) and let \( \{A_i\}_{i \in I} \) be the set of all convex \( l \)-subgroups of \( A \) such that \( x \notin A_i \). Denote \( B = \bigvee_{i \in I} A_i \). The fact that the system of all convex \( l \)-subgroups of a linearly ordered group is linearly ordered and 3.1 imply that \( B \) is the unique value of \( x \) in \( A \).

We set \( \{x\}^\delta = C, B + C = D \). Clearly \( C = A^\delta \). Hence we obtain by a routine calculation that \( D \) is a convex \( l \)-subgroup of \( G \). Moreover, \( D \) is a direct sum of its \( l \)-subgroups \( B \) and \( C \), and \( B \lor C = D \) is valid in the lattice \( c(G) \). We also have \( x \notin D \).

Let \( D_1 \) be a convex \( l \)-subgroup of \( G \) with \( x \notin D_1 \). Let \( 0 \leq d_1 \in D_1 \). Then \( x \leq d_1 \).

Let \( x \land d_1 = y, -y + d_1 = z, -y + x = y_1 \). We have \( z \geq 0, 0 < y_1 \leq x \) and \( y_1 \land z = 0 \). This and the fact that \( A \) is linearly ordered yields \( a \land z = 0 \) for each \( 0 \leq a \in A \). Thus \( z \in A^\delta \) and hence \( d_1 \in D \). Therefore \( D_1 \subseteq D \), which completes the proof.

If \( I \) is a linearly ordered set and if \( G_i \) is a linearly ordered group for each \( i \in I \), then \( \Gamma_{i \in I} G_i \) denotes the lexicographic product of the system \( \{G_i\} (i \in I) \) (cf., e.g., Fuchs [2]).

Let \( N \) be the set of all positive integers and let \( P = \{p_n\} (n \in N) \) be the set of all primes. Further, let \( R_0 \) be the set of all rational numbers (with the natural linear order).

Let \( f \) be a one-to-one mapping of the set \( R_0 \) onto \( N \) and let \( R_1, R_2 \) be infinite subsets of \( R_0 \) such that (i) \( R_1 \cap R_2 = \emptyset \), \( R_1 \cup R_2 = R_0 \), and (ii) both \( R_1 \) and \( R_2 \) are dense subsets of \( R_0 \). For each \( x \in R_0 \) let \( K_x \) be the set of all rational numbers of the form \( \lfloor p_m^n \rfloor \), where \( n = f(x), m \in N \) and \( l \) is any integer. We consider \( K_x \) as an additive group with the natural linear order. If \( x, y \in R_0 \) are distinct, then the linearly ordered groups \( K_x \) and \( K_y \) fail to be isomorphic. We denote by \( H_0 \) the class of all lattice ordered groups \( H \) that can be expressed as

\[ H = \Gamma_{i \in I} H_i, \]

where

(i) \( I \) is a convex subset of \( R_0 \);

(ii) for each \( i \in I, H_i \) is isomorphic with \( K_i \).
From the definition of $H_0$ it follows that if $K$ is a homomorphic image of a lattice ordered group $H$ belonging to $H_0$ then either $K$ belongs to $H_0$ or $K = \{0\}$. The same is valid for each convex $l$-subgroup of $H$.

Let $H \in H_0$ be as in (3) and let $0 < g \in H$. Let us denote by $i_0$ the least $i \in I$ with $g(i) \neq 0$. If $i_0 \in R_i$ ($i \in \{1, 2\}$), then the element $g$ will be said to be of type $R_i$. Let $R_i(H)$ be the set of all elements of $H$ which are of type $R_i$ ($i = 1, 2$). We have $R_1(H) \cap R_2(H) = 0$. If $\phi$ is an isomorphism of $H$ onto a linearly ordered group $H' \in H_0$, then $\phi(R_i(H)) = R_i(H')$ ($i = 1, 2$).

An isomorphism $\phi$ of a lattice ordered group $G_1$ into a lattice ordered group $G_2$ is said to be convex if $\phi(G_1)$ is a convex $l$-subgroup of $G_2$. Let $G$ be a lattice ordered group and $0 < x \in G$. The element $x$ will be called of type $R_1$ if there exist $H \in H_0$ and a convex isomorphism $\phi$ of $H$ into $G$ such that $x \in \phi(H)$ and $\phi^{-1}(x) \in R_i(H)$. Let $R_1(G)$ be the set of all elements of $G$ which are of type $R_1$. The set $R_2(G)$ is defined analogously. Then $R_1(G) \cap R_2(G) = 0$ is valid. Moreover, 3.2 implies that each element $x \in R_1(G) \cup R_2(G)$ possesses a unique value $v_0(x)$ in $G$. We put

$$s_1(G) = \{v_0(x) : x \in R_1(G)\}, \quad s_2(G) = \{v_0(x) : x \in R_2(G)\}.$$ 

3.3. Lemma. The mappings $s_1$ and $s_2$ fulfil the condition (1).

Proof. Let $G$ be a lattice ordered group and let $G_1$ be a convex $l$-subgroup of $G$. We have to verify that $s_1(G_1) = \{C \cap G_1 : C \in s_1(G) and C \not\subset G_1\}$.

Let $C \in s_1(G_1)$. There is $x \in R_1(G_1)$ such that $C = v_0(x)$. Let $B$ be the convex $l$-subgroup of $\{x\}^{\delta(G_1)}$ that is maximal with respect to the property of non-containing $x$; i.e., $B$ is the value of $x$ in $\{x\}^{\delta(G_1)\delta(G_1)}$. Then $B$ is also the value of $x$ in $\{x\}^{\delta}$. From 3.2 it follows that

$$C = v_0(x) = B + \{x\}^{\delta(G_1)}.$$

Further, we have $x \in R_1(G)$. Thus $x$ has a unique value in $G$; let us denote this value by $C = v_0(x)$. Then $C \in s_1(G)$, $C \subset G_1$ and by using 3.2 again we obtain

$$C = B + \{x\}^{\delta}.$$ 

Since $\{x\}^{\delta(G_1)} = \{x\}^{\delta} \cap G_1$, we get $C = C \cap G_1$. Thus $s_1(G_1) \subseteq \{C \cap G_1 : C \in s_1(G) and C \not\subset G_1\}$.

Now let $C \in s_1(G)$ such that $C \not\subset G_1$. There is $x \in R_1(G)$ with $C = v_0(x)$. Let $B$ be the value of $x$ in $\{x\}^{\delta}$. Then $C = B + \{x\}^{\delta}$. We shall show that $x \in G_1$.

By way of contradiction, assume that $x$ does not belong to $G_1$. From $C \not\subset G_1$ it follows that there exists $0 < g_1 \in G_1$ such that $g_1 \not\in C$. If $g_1 \geq x$, then $x \in G_1$, which is a contradiction. If $0 < z \in G$ and $z \leq x$, then the structure of lattice ordered groups belonging to $H_0$ yields that either $z \in B$ or the value of $z$ in $\{x\}^{\delta}$ coincides with $B$. If $g_1 < x$, then $g_1 \not\in B$ (because $g_1 \not\in C$) and thus the value of $g_1$ in $\{x\}^{\delta}$ coincides with $B$; but in this case there is a positive integer $n$ with $ng_1 > x$, implying $x \in G_1$. 

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Hence we can suppose that \( g_1 \) is incomparable with \( x \). Put \( y = x \land g_1, z = -y + + g_1 \). Then \( y \in B \) and \( z \in \{x\}^{\delta} \), hence \( g_1 \in C \), which is a contradiction. Therefore \( x \in G_1 \) and so \( B \subseteq G_1 \).

The relation \( x \in R_1(G) \cap G_1 \) implies \( x \in R_1(G_1) \). Thus

\[
C \cap G_1 = (B + \{x\}^{\delta}) \cap G_1 = (B \lor \{x\}^{\delta}) \cap G_1 = (B \land G_1) \lor (\{x\}^{\delta} \land G_1) = B \lor (\{x\}^{\delta} \land G_1) = B \lor \{x\}^{\delta(G_1)} = B + \{x\}^{\delta(G_1)} = v_{G_1}(x) \in s_1(G).
\]

We have proved that \( s_1 \) fulfils (1). The same proof can be applied to \( s_2 \).

3.4. Lemma. The mappings \( s_1 \) and \( s_2 \) fulfill the condition (2).

Proof. Let \( K \) be an \( l \)-ideal of a lattice ordered group \( G \) and let \( C \in s_1(G), C \supseteq K \).

We have to verify that \( C/K \) belongs to \( s_1(G/K) \).

According to the assumption there exists \( x \in R_1(G) \) such that \( C = v_G(x) \). As above, put \( A = \{x\}^{\delta}, B = v_A(x) \). For each \( y \in G, Y \subseteq G \) put \( \tilde{y} = y + K, \tilde{Y} = \{y + K\}_{y \in Y} \). The structure of \( A \) yields that the lattice ordered group \( \tilde{A} \) belongs to \( H_0 \) (the case \( \tilde{A} = \{0\} \) is impossible because \( \bar{x} \in \tilde{A} \) and \( \bar{x} \notin K \)); moreover \( \bar{x} \in R_1(\tilde{A}) \) and \( \bar{B} = v_A(\bar{x}) \). Thus \( \bar{x} \in G \).

Put \( D = \{x\}^{\delta} \). From 3.2 it follows that \( \bar{C} = \bar{B} + \bar{D} \). Hence in order to prove that \( \bar{C} = v_G(\bar{x}) \) it suffices to verify that

\[
\bar{D} = \{\bar{g} \in \bar{G} : |\bar{g}| \land \bar{x} = \bar{0}\},
\]

the symbol \( \bar{0} \) denoting the zero element in \( \bar{G} \).

If \( \bar{g} \in \bar{D}, \) then there is \( g_1 \in \bar{g} \cap D, \) hence \( |\bar{g}| \land \bar{x} = |\bar{g}_1| \land \bar{x} = |\bar{g}_1| \land \bar{x} = \bar{0} \). Conversely, suppose that \( \bar{g} \in \bar{G} \) and that \( |\bar{g}| \land \bar{x} \bar{0} \) is valid. There exists \( 0 \leq g_2 \in |\bar{g}| = |\bar{g}|. \) We have \( \bar{g}_2 \land \bar{x} = \bar{0}, \) hence \( 0 \leq z = g_2 \land x \in K. \) Put \( g_3 = -z + + g_2, x_1 = -z + x. \) Clearly \( x \notin K, \) thus \( 0 < x_1 \leq x. \) Moreover, we have \( g_3 \land x_1 = = 0. \) This and the fact that \( \{0, x\} \) is a chain imply \( g_3 \land x = 0. \) Hence \( g_3 \in D \) and therefore \( |\bar{g}| = \bar{g}_3 \in \bar{D} \). Thus \( \bar{g} \in \bar{D}, \) which completes the proof for \( s_1. \) The proof for \( s_2 \) is analogous.

From 3.3 and 3.4 we obtain:

3.5. Lemma. \( s_1 \) and \( s_2 \) are value selectors.

4. THE MAPPINGS \( s_1' \) AND \( s_2' \)

In this paragraph we shall use the same notation as in § 3. Let \( R_{01} \) be the class of all lattice ordered groups \( H \) such that \( H \) is isomorphic to some \( K_t, t \in R_1. \) The class \( R_{02} \) is defined analogously. We put \( R_0 = R_{01} \cup R_{02}. \)
Let \( G \in \mathcal{G}, \quad 0 < x \in G \). If there exists a convex \( l \)-subgroup \( H \) of \( G \) with \( x \in H \) such that \( H \) belongs to \( R'_{01} \), then the element \( x \) is said to be of type \( R_{01} \). The elements of type \( R_{02} \) or \( R_0 \), respectively, are defined analogously. Let \( R_{01}(G) \) be the set of all elements of \( G \) which are of type \( R_{01} \). Similarly we define the sets \( R_{02}(G) \) and \( R_0(G) \). According to 3.2, each element \( x \in R_0(G) = R_{01}(G) \cup R_{02}(G) \) possesses a unique value \( v_0(x) \) in \( G \). Put

\[
\sigma_0(G) = \left\{ v_0(x) : x \in R_0(G) \right\}, \quad \sigma_0(G) = \left\{ v_0(x) : x \in R_0(G) \right\}.
\]

**4.1. Lemma.** \( s_{01}, s_{02}, \) and \( s_0 \) are value selectors.

The proof is analogous to that used in § 3 for \( s_1 \) and \( s_2 \), and therefore will be omitted.

Put \( s'_i = s_i \cup s_0 \) for \( i = 1, 2 \) (i.e., \( s'_i(G) = s_i(G) \cup s_0(G) \) for each \( G \in \mathcal{G} \)). From 3.5 and 4.1 we obtain

**4.2. Lemma.** \( s'_1 \) and \( s'_2 \) are value selectors.

Let us denote by \( A_0 \) the class of all lattice ordered groups \( G \) such that either \( G = \{0\} \) or \( G \) is a direct sum (= discrete direct product) of lattice ordered groups belonging to \( R_0' \). Similarly we define the classes \( A_1 \) and \( A_2 \). It is easy to verify that all these classes are torsion classes (this follows also immediately from [9], Thm. 2.6).

Put \( B_i = T(s'_i) \). For each \( K_i \in R_0' \) we have \( s_0(K_i) = \{ \{0\} \} = M_0(K_i) \), hence \( K_i \in T(s_0) \subseteq T(s'_1) \). Because each lattice ordered group \( G \in A_0 \) is a join of lattice ordered groups belonging to \( R_0' \) and since \( T(s'_1) \) is a torsion class (cf. 2.1) we infer that

\[
A_0 \subseteq T(s'_1)
\]

is valid.

For each \( G \in \mathcal{G} \) we denote by \( A_0(G) \) the join of all convex \( l \)-subgroups of \( G \) which belong to \( A_0 \). Then \( A_0(G) \) belongs to \( A_0 \) as well.

**4.3. Lemma.** \( s_1(G) \cap s_2(G) = \emptyset \).

**Proof.** By way of contradiction, assume that \( C \in s_1(G) \cap s_2(G) \). According to 3.2 there exists \( 0 < x \in R_1(G), \quad 0 < y \in R_2(G), \quad B_1 \in c(G), \quad B_2 \in c(G) \) such that

\[
C = B_1 + \{x\}^\delta, \quad C = B_2 + \{y\}^\delta,
\]

where \( B_1 \) is the value of \( x \) in \( \{x\}^\delta \) and \( B_2 \) is the value of \( y \) in \( \{y\}^\delta \). Since \( R_1(G) \cap \cap R_2(G) = \emptyset \) we have \( x \neq y \). If \( x < y \), then \( x \in B_2 \subseteq C \), which is impossible; similarly, \( y \not< x \). Hence \( x \) is incomparable with \( y \); because \( [0, x] \) and \( [0, y] \) are chains, it follows that \( x \wedge y = 0 \), and thus \( y \in \{x\}^\delta \subseteq C \), which is a contradiction.

**4.4. Lemma.** Let \( y \in R_2(G), \quad y \notin R_{02}(G) \). Then \( v_0(y) \notin s_0(G) \).
Proof. Clearly \( s_{01} \leq s_1 \), hence 4.3 implies \( s_{01}(G) \cap s_2(G) = \emptyset \). Because of \( v_0(y) \in s_2(G) \) we have to verify that \( v_0(y) \notin s_{02}(G) \).

By way of contradiction, assume that \( v_0(y) \in s_{02}(G) \). Hence there exists \( z \in R_{02}(G) \) such that \( v_0(y) = v_0(z) \). From the structure of lattice ordered groups belonging to \( H_0 \) we infer that we have neither \( z = y \) nor \( z > y \). The cases (i) \( y > z \) and (ii) \( y \) is incomparable with \( z \) lead to a contradiction in a similar way as in the proof of 4.3.

4.5. Lemma. Let \( G \in B_1, \ C \in M_0(C) \). Then there is \( x \in R_0(G) \) such that \( C = v_0(x) \).

Proof. By way of contradiction, assume that \( C \neq v_0(x) \) for each \( x \in R_0(G) \). Then there is \( y \in R_1(G) \setminus R_{01}(G) \) such that \( C = v_0(x) \). Now the definition of \( H_0 \) implies that there is \( y \in R_2(G) \setminus R_{02}(g) \) with \( y < x \) (we use the density of \( R_2 \) in \( R_0 \)). From 4.3 and 4.4 we obtain \( v_0(y) \notin s^4(G) \) implying \( G \notin B_1 \), which is a contradiction.

4.6. Lemma. Let \( G \) belong to \( B_1 \). Then \( G = A_0(G) \).

Proof. Suppose that \( G \neq A_0(G) \). Then there is \( y \in G \setminus A_0(G) \). There exists a value \( C \) of \( y \) in \( G \) such that \( A_0(G) \subseteq C \). In view of 4.5, there is \( x \in R_0(G) \) with \( C = v_0(x) \). The convex \( l \)-subgroup \( C_1 \) of \( G \) generated by \( x \) belongs to \( A_0 \), hence \( x \in C_1 \subseteq A_0(G) \subseteq C \), which is a contradiction.

From (4) and 4.6 we conclude

4.7. Lemma. \( T(s'_1) = A_0 \).

Analogously we obtain

4.8. Lemma. \( T(s'_2) = A_0 \).

4.9. Lemma. Let \( H \) be as in (3) with \( I = R_0 \). Then \( H \in T(s'_1 \lor s'_2) \) and \( H \neq A_0 \).

Proof. If \( C \) is a value in \( H \), then there is \( 0 < x \in H \) such that \( C \) is a value of \( x \). Since \( x \) belongs either to \( R_1(H) \) or to \( R_2(H) \), \( C \) belongs to \( \langle s'_1 \lor s'_2 \rangle (H) \). Hence \( H \in T(s'_1 \lor s'_2) \). Moreover, \( H \) is linearly ordered and thus \( H \) is directly indecomposable. Hence from \( H \neq R'_0 \) it follows that \( H \) does not belong to \( A_0 \).

4.10. Corollary. There does not exist any largest value selector \( M \) with \( T(M) = A_0 \).

Hence the above questions quoted from [7] are answered by the following

Proposition. The function \( M \rightarrow T(M) \) does not, in general, preserve joins. If \( A \) is a torsion class, then there need not exist a largest value selector \( M \) with \( T(M) = A \); moreover, the class of all value selectors \( M_1 \) with \( T(M_1) = A \) need not be directed.
References


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