Václav Alda
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ON SEGAL'S POSTULATES FOR GENERAL QUANTUM MECHANICS

VÁCLAV ALDA, Praha

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Segal's postulates [1] deal with real algebraic systems. There are two groups of postulates.

I.1. The system $\mathcal{H}$ is a real linear space.

2. In $\mathcal{H}$ there exist an identity element $I$ and for every $U \in \mathcal{H}$ and a positive integer $n$ an element $U^n$ of $\mathcal{H}$, such that usual rules for operations with polynomials in a single variable are valid: if $f, g$ and $h$ are polynomials with real coefficients, and if $f(g(x)) = h(x)$ for all real $x$, then $f(g(U)) = h(U)$; here $f(U) = \beta_0 I + \sum_{k=1}^{m} \beta_k U^k$ if $f(x) = \sum_{k=0}^{m} \beta_k x^k$.

II.1. $\mathcal{H}$ is a real Banach space with the norm $\| \|$.

2. $\| U^2 - V^2 \| \leq \max [\| U^2 \|, \| V^2 \|]$.

3. $\| U^2 \| = \| U \|^2$.

4. $\| \sum_{U \in \mathcal{H}} U^2 \| \leq 2 \sum_{U \in \mathcal{S}} U^2 \|$ if $\mathcal{H} \subseteq \mathcal{S}$ and $\mathcal{S}$ is a finite subset of $\mathcal{H}$.

5. $U^2$ is a continuous function of $U$.

The reality of $\mathcal{H}$ is expressed in II.2 and II.4. Sherman [2] proved that II.4 is redundant as it is a consequence of the other postulates — he in fact showed that the sum of squares is a square and this, by Corollary 1 of [1] (II.4 is not needed for its proof), implies the desired result.

However, this can be also seen directly.

$$\| U^2 \| = \| (U^2 + V^2) - V^2 \| \leq \max (\| U^2 + V^2 \|, \| V^2 \|)$$

as the sum of squares is a square and so II.2 can be used. If we suppose $\| U^2 \| > \| V^2 \|$, then $\| U^2 \| \leq \| U^2 + V^2 \|$. If $\| U^2 \| = \| V^2 \|$ then we can write

$$\| U^2 \| = \| (U^2 + tV^2) - tV^2 \|, \ 0 < t < 1,$$
and thus \( \|U^2\| \leq \|U^2 + tV^2\| \). As \( \|U^2 + V^2 - (U^2 + tV^2)\| = (1 - t) \|V^2\| \to 0 \) for \( t \to 1 \), we have \( \|U^2\| \leq \|U^2 + V^2\| \) as well.

**Remark.** When proving that the sum of squares is a square \([2]\), the author uses the series for \( \sqrt{1 - t} \). Now, we must know how the values of \( \|U^n\| \) are distributed for using the series for \( \sqrt{(I - U)} \).

If we have II.2, then we can calculate

\[
U^{n+1} = \frac{1}{2}(U^n + U)^2 - (U^2 - U)^2
\]

and consequently, if \( \|U\| \leq 1 \), then, by induction, \( \|U^{n+1}\| \leq 1 \).

If we have II.4, then we have to use the inequality \( \|U^n\| \leq 2\|U^{n-1}\| \cdot \|U\| \) (see below) for evaluating \( \|U^n\| \) and so \( \|U^{n+1}\| \leq 2^n \) for \( \|U\| \leq 1 \).

In \([2]\) only a non-vanishing radius of convergence was used for the proof.

End of the remark.

We shall show now:

For a commutative system of observables \( \mathcal{A} \) the positivity of squares as expressed in II.4 is sufficient for the demonstration of Theorem 1 in \([1]\) and so II.2 is a consequence of II.4 (for commuting observables).

The product in \( \mathcal{A} \) is \( x \circ y = \frac{1}{2}((x + y)^2 - (x - y)^2) \) and so \( 4\|x \circ y\| = 4\|(x + y)^2 - (x - y)^2\| \leq \|(x + y)^2\| + \|(x - y)^2\| = \|x + y\|^2 = \|x - y\|^2 \) by II.3.

Hence for \( \|x\|, \|y\| \leq 1 \) we have \( 4\|x \circ y\| \leq 8 \) and thus \( \|x \circ y\| \leq 2\|x\| \cdot \|y\| \) for all \( x, y \).

If we set \( \|x\| = 2\|x\| \) as a new norm, \( \mathcal{A} \) will be a real Banach algebra. In this new norm, we have

\[
\|x\|^2 = 2\|x\|^2.
\]

Let \( \mathcal{A}_c \) be the complexification of \( \mathcal{A} \):

\[
\mathcal{A} = \{z \mid z = x + iy, x, y \in \mathcal{A}\}.
\]

For \( z = x + iy \) we set \( \|z\| = \|x\| + \|y\|, z^* = x - iy \). Then \( \|z + \zeta\| \leq \|z\| + \|\zeta\| \), \( |az| = |a| \cdot |z| \) for a real \( a \),

\[
|z\zeta| = |x\xi - y\eta| + \ldots \leq |x| \cdot |\xi| + \ldots = |z| \cdot |\zeta|.
\]

Finally,

\[
\|z\|^2 = (\|x\|^2 + \|y\|^2) = \|x\|^2 + \|y\|^2 + 2|x| \cdot |y| \leq 2\|x\|^2 + 2\|x\|^2 = 4\|x\|^2 = 8\|x\|^2 \leq 8\|x\|^2 + 2\|x\|^2 = 8\|x\|^2 + y^2 \text{ for } |x| \geq \|y\|.
\]

On the other hand,

\[
\|zz^*\| = |x^2 + y^2|, \text{ hence } \|z\|^2 \leq 8\|zz^*\|.
\]

If we set

\[
N(z) = \sup_{0 \leq \alpha \leq 2\pi} |\exp(i\alpha) z|,
\]

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we have
\[ N(z + \zeta) \leq N(z) + N(\zeta), \quad N(az) = |a| N(z), \]
\[ N(z \zeta) \leq N(z) N(\zeta), \]
and
\[ N^2(z) \leq 8 N(zz^*). \]

Now we shall apply the result of 26.E from [3]. \( \mathfrak{H}_C \) with the norm \( N \) is \(*\)-algebraically isomorphic to \( C(\mathfrak{M}) \) — the space of continuous functions on a compact — and we have
\[ |\hat{\xi}|_\infty \leq N(z) \leq 8|\hat{\xi}|_\infty, \quad z \in \mathfrak{H}_C, \]
where \( \hat{\xi} \) is the corresponding function and \( |\cdot|_\infty \) is the norm in \( C(\mathfrak{M}) \).

Now \( N(x) = \sqrt{2} \|x\| \) for \( x \in \mathfrak{H} \) and hence
\[ 2^{-1/2}|\xi|_\infty \leq \|x\| \leq 4 \sqrt{2} |\xi|_\infty. \]

By II.3, \( \|x^{2k}\| = \|x\|^{2k} \) in \( \mathfrak{H} \) and the same is true in \( C(\mathfrak{M}) : (\xi^{2k})_\infty = |\xi|^{2k}_\infty. \)

Hence it must be \( \|x\| = |\xi|_\infty \) and this is the rest of Theorem 1.

Remark. The proof works well with the inequality
\[ A\|U^2\| \leq \|U\|^2 \leq B\|U^2\| \quad \text{for all} \quad U \in \mathfrak{H} \]
and \( \log (\|U\|^k/\|U^k\|) \) bounded for every \( U \) and a sequence \( k \to \infty \) instead of II.3.

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References


Author’s address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).