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NON-COMMUTATIVE INTERPOLATION OF SOBOLEV-BESOV
AND LEBESGUE SPACES WITH WEIGHTS

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(Received October 4, 1976)

The paper contains results about interpolation of weighted Sobolev-Besov and
Lebesgue spaces in domains. Further, we give examples and counter-example of
non-commutative interpolation. These results arise from the problem of finding
conditions ensuring

\[ F(\{A_0, A_1 \cap A_2\}) = F(\{A_0, A_1\}) \cap F(\{A_0, A_2\}), \]

where \( A_0, A_1 \) and \( A_2 \) are Banach spaces contained in a linear Hausdorff space and \( F \)
is an interpolation functor. Results of the type (\( \ast \)) are well-known if the so-called
commutative interpolation is considered. The first result may be found in Lions [8].
Later, Peetre [12] showed that (\( \ast \)) is true using the theory of quasilinearisable inter-
polation couples; he supposed that the operators involved commute with each other.
The interpolation of domains of infinitesimal generators of commuting semi-groups
of bounded operators or the interpolation of domains of positive operators with
commuting resolvents are special cases of this result. All these assertions are based
on hypotheses of commutativity and were mainly obtained by the authors mentioned
above and by T. Muramatu [9], H. Komatsu [7] and by P. Grisvard [3]. These
statements are contrasted not only by counter-examples (see H. Triebel [15]), but
also by results of the type (\( \ast \)) which were obtained independently of the assertions
mentioned above; in particular, the commutativity assumptions are not fulfilled.
For details we refer to H. Triebel [13, 14, 15, 16]. So it is interesting to extend (\( \ast \))
results for the real interpolation functor.

In our work we will only make use of the very applicable result by P. Grisvard.
(\( \ast \)) is proved by him in the case that \( A_0 \) contains \( A_2 \) and \( A_1 \) is the domain of a positive
operator \( A \) acting in \( A_0 \), with a positive restriction on \( A_2 \) (see Section 1). On the one
hand, our paper applies the theory of non-commutative interpolation to function
spaces of Sobolev-Besov or Lebesgue types, respectively, with and without weights;
on the other hand, we want to give an idea of how far these methods can be used.
In our considerations, the Banach spaces $A_0$ and $A_2$ will be weighted or unweighted Sobolev-Besov or Lebesgue spaces, for instance: $A_0 = L_p(\Omega)$, $A_2 = B_{pp}^s(\Omega)$ (the well-known Besov space; if $s \neq \text{integer}$ then the Slobodeckij spaces arise). The positive operator $A$, acting in $A_0$, is the multiplication operator: $(Au)(x) = \phi(x)u(x)$ where $\phi(x)$ is a measurable positive function. Thus, the domain $D(A)$, of definition of $A$ is a weighted Sobolev-Besov space and for several classes of functions $\phi(x)$ (described by certain growth conditions) we get results of the type

$$(L_p(\Omega), L_p(\Omega, \phi) \cap B_{pp}^s(\Omega))_{\theta,q} =$$

$$= (L_p(\Omega), L_p(\Omega, \phi))_{\theta,q} \cap (L_p(\Omega), B_{pp}^s(\Omega))_{\theta,q}.$$ 

With the aid of non-commutative interpolation we get, for a class of weight-functions $\phi_0(x)$ and $\phi_1(x)$,

$$(W_p^m(\Omega, \phi_0), W_p^m(\Omega, \phi_1)) \cap W_p^l(\Omega, \phi_0))_{\theta,p} =$$

$$= (W_p^m(\Omega, \phi_0), W_p^m(\Omega, \phi_1))_{\theta,p} \cap (W_p^m(\Omega, \phi_0), W_p^m(\Omega, \phi_0))_{\theta,p}.$$ 

Simultaneously we extend earlier results by A. Favini [1] about the interpolation 

$(W_p^m(\Omega, \phi_0), W_p^m(\Omega, \phi_1))_{\theta,p}$. 

Finally, we present a series of examples and counter-examples to ($\ast$).

1. MULTIPLICATION OPERATOR IN UNWEIGHTED SOBOLEV SPACES

First we describe the result by P. Grisvard [4]. We use the following statement which is important for the whole work:

**Proposition.** Let $A_0$ and $A_2$ be Banach spaces, $A_2 \subseteq A_0$. Let $A$ be a closed unbounded operator acting in $A_0$ with $D(A) = A_1$ (note that $A_1$ is a Banach space equipped with the norm $\|u\|_{A_1} = \|u\|_{A_0} + \|Au\|_{A_0}$, $u \in D(A)$). We assume that the operator $A$ satisfies the following conditions:

1. For $0 < t < \infty$ the operator $tA + E$ is invertible in $A_0$ and \begin{equation} \|(tA + E)^{-1}u\|_{A_0} \leq c_1\|u\|_{A_0}, \quad u \in A_0; \end{equation}

2. for $0 < t < \infty$ the operator $tA + E$ is invertible in $A_2$ and \begin{equation} \|(tA + E)^{-1}u\|_{A_2} \leq c_2\|u\|_{A_2}, \quad u \in A_2 \end{equation}

with constants $c_1$ and $c_2$ independent of $t$.

I take the opportunity to thank H. Triebel for pushing me on.

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Then for $0 < \theta < 1$, $1 < q < \infty$,

$$
(A_0, A_1 \cap A_2)_{\theta,q} = (A_0, A_1)_{\theta,q} \cap (A_0, A_2)_{\theta,q}
$$

is valid.

Here $(\cdot, \cdot)_{\theta,q}$ denotes the real interpolation functor (for instance the $K$-functor in the sense of J. Peetre [11]). We denote the conditions (1) and (2) in the proposition by “$A_0$-condition” and “$A_2$-condition”, respectively, and state:

If, for the closed operator $tA + E; 0 < t < \infty$, there exists a suitable operator $I_t: A_0 \to A_0; 0 < t < \infty$, such that

(4) for $0 < t < \infty$, $tA + E$ is an operator from $D(A)$ onto $A_0$;

(5) the operator $tA + E; 0 < t < \infty$, is a bounded operator from $A_0$ into $A_0$;

(6) in $A_0$, $I_t(tA + E) = E$ holds,

then the operators $tA + E; 0 < t < \infty$, are invertible in $A_0$: $(tA + E)^{-1} = I_t$.

If, additionally, (1) is valid, then the $A_0$-condition is fulfilled. The same can be said about the $A_2$-condition. Replacing the operators $I_t$ by $I_t'$, and the space $A_0$ by $A_2$ in (4), (5) and (6) we get the properties $(4')$, $(5')$ and $(6')$, respectively.

**Basic notations.** Let $\Omega$ be an arbitrary domain in $R_n$ with a boundary $\partial \Omega$ and a closure $\overline{\Omega}: \partial \Omega = \overline{\Omega} - \Omega$. $L^\infty(\Omega)$ denotes the set of all functions, Lebesgue-measurable in $\Omega$ with bounded essential suprema. By $W^m_p(\Omega); m = 1, 2, \ldots; 1 < p < \infty$, we denote the usual Sobolev spaces equipped with the norms

$$
\|u\|_{W^m_p(\Omega)} = \left( \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}, \quad u \in W^m_p(\Omega).
$$

Let $C_0^\infty(\Omega)$ be the set of all infinitely differentiable finite functions; we denote the closure of this set in the Banach space $W^m_p(\Omega)$ by $W^m_p(\Omega)$. Denote by $L^p(\Omega, \varphi(x))$ the usual weighted space with a non-negative measurable function $\varphi(x)$. The norm is given by

$$
\|u\|_{L^p(\Omega, \varphi(x))} = \int\varphi(x) u(x)^p \, dx^{1/p}, \quad u \in L^p(\Omega, \varphi(x)).
$$

The spaces arising by the real interpolation of Sobolev spaces $W^m_p(\Omega)$ are the Besov spaces $B^m_p,q(\Omega)$; by the complex interpolation we have the Lebesgue (Liouville or Bessel-potential) spaces $H^m_p(\Omega)$. We introduce a smoothness-property for bounded domains in $R_n$:

**Definition.** A bounded domain $\Omega$ belongs to a class $C^m; m = 1, 2, \ldots$ or $m = \infty$, if the following conditions are satisfied:
(9) There exist balls $K_j, j = 1, \ldots, N$, with
\[ \bigcup_{j=1}^{N} K_j \supseteq \partial \Omega \quad \text{and} \quad K_j \cap \partial \Omega = \emptyset. \]

(10) There exist vector-functions $f^{(j)}(x) = (f_1^{(j)}(x), \ldots, f_n^{(j)}(x))$ defined on $\overline{K}_j$ and satisfying the following conditions:

(i) The functions $f^{(j)}_i(x)$ are continuously differentiable up to the $m$-th order.

(ii) $y = f^{(j)}(x)$ is a one-to-one map from $K_j$ onto a bounded domain in $\mathbb{R}_n^+$. $f^j$ maps $\partial \Omega \cap K_j$ onto a part of the domain in $\mathbb{R}_n \{y : y \in \mathbb{R}_n; y_n = 0\}$ and $\Omega \cap K_j$ onto a simple-connected domain in the half-space $\mathbb{R}_n^+$.

(iii) $\frac{\partial(f^{(j)}_1, \ldots, f^{(j)}_n)}{\partial(x_1, \ldots, x_n)} \equiv 0$ for $x \in \overline{K}_j$.

$f^{(j)}$ denotes the transformation inverse to $f^{(j)}$.

Remark. If $\Omega$ is a domain belonging to $C^m$, then there exists a compact domain $\omega, \overline{\omega} \subset \Omega$, with
\[ \Omega \subset \bigcup_{j=1}^{N} K_j \cup \omega. \]

Domains $\Omega$ belonging to a class $C^m$ are called domains with a $C^m$-boundary.

1.1. THE CLASSES $K_1^{(m)}$ OF WEIGHT-FUNCTIONS

A weight-function is a function $\varphi(x) : \Omega \to \mathbb{R}_1$ which is measurable, positive and infinitely differentiable.

Definition. A weight-function $\varphi(x)$ belongs to a class $K_1^{(m)}; m \in \{1, 2, \ldots\}$, if:

(1) $\frac{1}{\varphi(x)} \in L^\infty(\Omega), \quad \frac{D^x \varphi(x)}{\varphi(x)} \in L^\infty(\Omega), \quad 0 < |x| \leq m.$

The classes $K_1^{(m)}$ are closed with respect to addition, multiplication and involution. Now we specify the Banach spaces $A_0, A_1$ and $A_2$ is Grisvard's proposition:

We assume that $\Omega$ is an arbitrary domain in $\mathbb{R}_n, m$ is an integer and $\varphi(x)$ is a weight-function. We set:

(3) $A_0 = L^p(\Omega), \quad A_2 = W^m_p(\Omega)$
and
\[ A_0 = L_p(\Omega), \quad A_2 = W_p^m(\Omega), \]
respectively. In both cases we put:
\[
(Au)(x) = \varphi(x)u(x), \quad D(A) = \{ u : u \in L_p(\Omega), \varphi u \in L_p(\Omega) \},
\]
\[
A_1 = [D(A), \| \cdot \|_\alpha] = L_p(\Omega, \varphi(x)).
\]

**Theorem.** Let \( \Omega \) be an arbitrary domain in \( \mathbb{R}_n \). Let the Banach spaces \( A_0, A_1 \) and \( A_2 \) be given by (3).

For weight-functions \( \varphi(x) \) belonging to \( K_1^{(m)} \) the conditions of Grisvard's proposition are satisfied and for \( 0 < \theta < 1, 1 < p < \infty \) we have
\[
(L_p(\Omega), W_p^m(\Omega) \cap L_p(\Omega, \varphi(x)))_{\theta,p} = W_p^m(\Omega) \cap L_p(\Omega, \varphi(x)^\theta).
\]

**Proof.** We prove the \( A_0 \)-condition. First we show 1. (1), (4), (5), (6). For \( 0 < t < \infty \) we put
\[
(I_t u)(x) = \frac{1}{t \varphi(x) + 1} u(x), \quad D(I_t) = L_p(\Omega).
\]

1. (1), (4), (5), (6) are obvious, since
\[
\int_\Omega \left| \frac{u}{t \varphi + 1} \right|^p dx \leq c \int_\Omega |u|^p dx, \quad c \text{ independent of } t,
\]
and
\[
\int_\Omega \left( \frac{\varphi}{t \varphi + 1} \right)^p |u|^p dx \leq c(t) \int_\Omega |u|^p dx
\]
are valid.

Thus, the \( A_0 \)-condition is satisfied with \((tA + E)^{-1} = I_t\).

Secondly we prove the \( A_2 \)-condition. We show 1. (2), (4'), (5'), (6'). For \( 0 < t < \infty \) we put
\[
(I'_t u)(x) = \frac{1}{t \varphi(x) + 1} u(x), \quad D(I'_t) = W_p^m(\Omega).
\]

Using the formula
\[
D^\varphi \left( \frac{1}{t \varphi + 1} \right) =
\]
\[
= \sum_{k_1 \alpha_1 + \ldots + k_n \alpha_n = [n]} \frac{\varphi^{k_1 + \ldots + k_n}}{(t \varphi + 1)^{k_1 + \ldots + k_n + 1}} \left( \frac{\partial^{\alpha_1} \varphi}{\partial x_1^{\alpha_1}} \right)^{K_1} \cdots \left( \frac{\partial^{\alpha_n} \varphi}{\partial x_n^{\alpha_n}} \right)^{K_n}
\]

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\((K_1, \ldots, K_n; \ a_1, \ldots, a_n \text{ non-negative integers})\) we get

\[
\left(10\right) \quad \int_{\Omega} \left| \frac{u}{t^{\varphi} + 1} \right|^p \, dx + \sum_{0 < |x| \leq m} \int_{\Omega} \left| D^2 \left( \frac{u}{t^{\varphi} + 1} \right) \right|^p \, dx \leq c \left[ \int_{\Omega} |u|^p \, dx + \sum_{0 < |x| \leq m} \sum_{\beta_1 + \beta_2 = \alpha} \int_{\Omega} \left| D^{\beta_1} \left( \frac{1}{t^{\varphi} + 1} \right) D^{\beta_2} u \right|^p \, dx \right] \leq c \|u\|_{W^{m}(\Omega)}^{p}
\]

for \(u \in W^{m}(\Omega)\), where \(c\) is a positive constant independent of \(t\). Similarly,

\[
\left(11\right) \quad \int_{\Omega} \left| \frac{\varphi}{t^{\varphi} + 1} u(x) \right|^p \, dx + \sum_{0 < |x| \leq m} \int_{\Omega} \left| D^2 \left( \frac{\varphi}{t^{\varphi} + 1} u(x) \right) \right|^p \, dx \leq c(t) \|u\|_{W^{m}(\Omega)}^{p}
\]

follows. Thus, the \(A_2\)-condition is satisfied with \((tA + E)^{-1} = I\). Using

\[
\left(12\right) \quad (L_p(\Omega), L_p(\varphi(x)))_{\alpha,p} = L_p(\Omega, \varphi(x))^p
\]

and

\[
\left(13\right) \quad (L_p(\Omega), W^{m}(\Omega))_{\alpha,p} = W^{m \theta}(\Omega)
\]

(cf. H. Triebel [16] Ch. 1 and Ch. 4, resp.) we get (5). If \(l\) is an integer, \(1 \leq l < m\), then under the same conditions as in the Theorem, the identity

\[
(L_p(\Omega), W^{l}(\Omega) \cap L_p(\Omega, \varphi(x)))_{\alpha,p} = W^{l \theta}(\Omega) \cap L_p(\Omega, \varphi^\theta)
\]

holds.

**Remark 1.** An analogous theorem holds, if the Banach spaces \(A_0, A_1\) and \(A_2\) are specified in the sense of (4). For the spaces \((L_p(\Omega), W^{l}(\Omega))_{\alpha,p}, l \in \{1, 2, \ldots, m\}\), we refer to P. Grisvard [5] or to Section 4.

**Remark 2.** For \(\Omega = R_n\), the functions \(\varphi(x) = (1 + |x|^2)^p; \ \eta > 0\), are examples of weight-functions belonging to all classes \(K^{(m)}_1; m = 1, 2, 3, \ldots\).

### 1.2. The Classes \(K^{(m)}_2\) of Weight-Functions

Let \(\Omega\) be an arbitrary domain in \(R_n\).

**Definition.** A weight-function \(\varphi(x)\) belongs to a class \(K^{(m)}_2\), \(m \in \{1, 2, \ldots\}\), if:

\[
\left(1\right) \quad \frac{1}{\varphi(x)} \in L^{\eta}(\Omega),
\]

\[
\left(2\right) \quad \text{for all } 0 < |x| \leq m \text{ we have } D^2 \varphi|\varphi \in L_p, \text{ with } p_x = n/|x|.
\]

The classes \(K^{(m)}_2\) are closed with respect to addition, multiplication and involution.

**Theorem.** Let \(\Omega\) be an arbitrary domain in \(R_n\). The Banach spaces \(A_0, A_1\) and \(A_2\)
are specified in the sense of 1.1 (3). The weight-functions \( \varphi(x) \) belonging to \( K_2^{(m)} \) satisfy the conditions of Grisvard's proposition and for \( 0 < \theta < 1, 1 < p < \infty, \) 1.1 (5) holds.

The proof is obtained by Hölder's inequality and by the fact that for suitable real numbers \( p, l, q, k \) the spaces \( W^l_p(\Omega) \) are embedded in the spaces \( W^k_q(\Omega) \). More exactly, let \( \Omega \) be an arbitrary domain in \( R^n \). Then for \( 0 \leq t \leq s < \infty \) and \( \infty > q \geq p > 1, \)

\[
W^s_p(\Omega) \subseteq W^t_q(\Omega), \quad s - \frac{n}{p} \geq t - \frac{n}{q},
\]

is valid.

Remark 1. Using the statement

\[
\text{for } \frac{n}{|\alpha|} < q < \infty, \quad L_q(\Omega) \subseteq L_{n/|\alpha|}(\Omega) + L^\infty(\Omega) \quad \text{holds},
\]

we put Theorems 1.1 and 1.2 together as follows:

**Corollary.** Let \( \Omega \) be an arbitrary domain in \( R^n \). The spaces \( A_0, A_1 \) and \( A_2 \) are given by 1.1(3). Let \( \varphi(x) \) be a weight-function satisfying

\[
\frac{1}{\varphi(x)} \in L^\infty(\Omega),
\]

\[
\text{for every } \alpha, 1 \leq |\alpha| \leq m, \text{ there exists a number } q_\alpha,
\]

\[
\frac{n}{|\alpha|} < q_\alpha \leq \infty, \quad \text{such that } \frac{D^\alpha \varphi}{\varphi} \in L_{q_\alpha}(\Omega).
\]

Then Grisvard's-conditions are fulfilled and 1.1 (5) is valid.

Remark 2. Remark 1.1.1 again applies.

Remark 3. We give an example of a weight-function \( \varphi(x) \) belonging to all classes \( K_2^{(m)}, m = 1, 2, \ldots \)

We set \( \Omega = \{(x, y) : 1 < x < \infty; 0 < y < e^{-x}\} \) and \( \varphi(x) = e^{x^2} \). The function \( \varphi(x) \) is a weight-function. For multi-indices \( \alpha = (\alpha_1, \alpha_2) \) with \( \alpha_2 = 0 \) we have

\[
D^\alpha \varphi(x) = e^{x^2} P_{|\alpha|}(x) \quad \text{with } P_{|\alpha|} \text{ a polynomial in } x \text{ of degree } |\alpha|.
\]

Thus, for \( 1 < q < \infty, \)

\[
\int_{\Omega} \left| \frac{1}{\varphi(x)} D^\alpha \varphi \right|^q dx = \int_1^\infty \int_0^{e^{-x}} |P_{|\alpha|}(x)|^q y^q dy dx = \int_1^{\infty} |P_{|\alpha|}(x)|^q e^{-x} dx \leq \infty
\]

is valid, so that \( \varphi(x) \) belongs to all classes \( K_2^{(m)}, m = 1, 2, \ldots \). It is clear that \( \varphi(x) \) does not belong to any class \( K_1^{(m)}, m \in \{1, 2, \ldots\} \).
1.3. THE CLASSES $K^{(m)}_3$ OF WEIGHT-FUNCTIONS

Throughout this section $\Omega$ is a bounded domain in $R_n$ with a $C^\infty$-boundary. The properties of such domains gave rise to classes $K^{(m)}_3$, $m \in \{1, 2, \ldots\}$.

**Definition.** Let $\Omega$ be a bounded domain in $R_n$ with a $C^\infty$-boundary; $d(x)$ denotes the distance from the point $x \in \Omega$ to the boundary $\partial \Omega$. The class $K^{(m)}_3$, $m \in \{1, 2, \ldots\}$, consists of all weight-functions $\varphi(x)$ satisfying

\[
(1) \quad \frac{1}{\varphi(x)} \in L^\infty(\Omega),
\]

\[
(2) \quad \text{for all } 1 \leq |x| \leq m, \quad \frac{D^s \varphi}{\varphi} d(x)^{|s|} \in L^\infty(\Omega) \quad \text{holds}.
\]

The classes $K^{(m)}_3$ are closed with respect to addition, multiplication and involution. Next we give a statement without proof which we use throughout this section.

**Lemma** (Hardy’s inequality applied to domains). Let $\Omega$ be a bounded domain in $R_n$ with a $C^\infty$-boundary. We assume that a natural number $m$ and real numbers $\alpha$ and $p$, $1 < p < \infty$, satisfy $\alpha - mp + kp \neq -1$, $k = 0, \ldots, m - 1$. Then there exists a positive constant $c$ and a domain $\omega, \omega \subset \Omega$, such that

\[
(3) \quad \int_{\Omega} d(x)^{\alpha - mp} |u(x)|^p \, dx \leq c \int_{\Omega} d(x)^{\alpha} \sum_{|\gamma| = m} |D^\gamma u|^p \, dx + c \int_{\omega} |u|^p \, dx
\]

holds for all functions $u(x) \in \overline{C^\infty}(\Omega)$, for which the left integral in (3) converges. ($\overline{C^\infty}(\Omega)$ is defined as usual.)

**Remark 1.** We state a special case of this lemma:

Functions $u(x) \in \hat{W}^{m}_{p}(\Omega)$ satisfy the estimate

\[
(4) \quad \left\| \frac{u}{d(x)^m} \right\|_{L^p(\Omega)} \leq c \left\| u \right\|_{\hat{W}^{m}_{p}(\Omega)}.
\]

**Theorem.** Let $\Omega$ be a bounded domain in $R_n$ with a $C^\infty$-boundary. The Banach spaces $A_0$, $A_1$ and $A_2$ are specified by 1.1 (4). For weight-functions $\varphi(x)$ belonging to $K^{(m)}_3$ the conditions of Grisvard’s proposition are satisfied and

\[
(7) \quad (L^p(\Omega), \hat{W}^{m}_{p}(\Omega) \cap L^p(\Omega, \varphi))_{\theta, p} = (L^p(\Omega), \hat{W}^{m}_{p}(\Omega))_{\theta, p} \cap L^p(\Omega, \varphi),
\]

holds for $0 < \theta < 1$, $1 < p < \infty$.

**Proof.** The proof of the $A_0$-condition is similar to that in Section 1.1. To prove
the $A_2$-condition we use Remark 1 and formula 1.1 (9), thus obtaining the following estimate, for functions $u(x) \in \tilde{W}^m_p(\Omega)$:

\begin{equation}
\sum_{|\alpha| \leq m} \int_{\Omega} \left| D^\alpha \left( \frac{u}{tu + 1} \right) \right|^p dx \leq \nonumber \end{equation}

\begin{equation}
\leq c \left( \int_{\Omega} |u|^p dx + \sum_{0 < |\beta| \leq m} \sum_{\beta_1 + \beta_2 = \beta} \int_{\Omega} \left| D^{\beta_1} \left( \frac{1}{tu + 1} \right) D^{\beta_2} u \right|^p dx \right) \leq c' \| u \|_{W^m_p(\Omega)}, \nonumber \end{equation}

where $c'$ is a positive constant independent of $t$.

Analogously we estimate for the same $u$:

\begin{equation}
\sum_{|\alpha| \leq m} \left\| D^\alpha \left( \frac{\varphi}{tu + 1} \right) \right\|^p_{L_p(\Omega)} \leq c(t) \| u \|_{W^m_p(\Omega)} < \infty, \nonumber \end{equation}

where the constant $c$ depends on the parameter $t$.

Thus, the functions $u(x)/(t \varphi(x) + 1)$ and $(\varphi(x)/(t \varphi(x) + 1)) u(x)$ belong to $\tilde{W}^m_p(\Omega)$ so that Grisvard’s-conditions are satisfied. By means of 1.1 (12) we get (7). To determine the spaces $(L_p(\Omega), \tilde{W}^m_p(\Omega))_{\theta,p}$ we refer to P. Grisvard [5] or to Section 4 of our paper. There we state Grisvard’s result.

Remark 2 (Examples of weight-functions belonging to $K_3^{(m)}$). Let $\Omega$ be a bounded domain in $R_n$ with a $C^\infty$-boundary.

1. The functions $\varphi(x) = d(x)^\alpha$, $\alpha$ real, $\alpha \leq 0$, belong to all classes $K_3^{(m)}$, $m = 1, 2, \ldots$; they belong neither to any class $K_1^{(m)}$, $m = 1, 2, \ldots$, nor to any class $K_2^{(m)}$, $m = 1, 2, \ldots$.

We assume that $\sup d(x) < 1$. Then the functions $\varphi(x) = \ln (d(x)^\alpha)$, $\alpha < 0$, belong to all classes $K_3^{(m)}$ but not to any class $K_1^{(m)}$ or $K_2^{(m)}$, $m = 1, 2, \ldots$.

Remark 3. The space $\tilde{W}^m_p(\Omega)$ can be represented by the following intersection:

\begin{equation}
\tilde{W}^m_p(\Omega) = W^m_p(\Omega) \cap L_p(\Omega, d(x)^{-m}), \quad m = 1, 2, \ldots, \quad 1 < p < \infty. \nonumber \end{equation}

Setting $A_0 = L_p(\Omega), A_2 = W^m_p(\Omega)$ and $(A u)(x) = d(x)^{-m} u(x), D(A) = L_p(\Omega, d(x)^{-m})$, the $A_0$-condition is proved immediately. The weight-function $d(x)^{-m}$ belongs to the class $K_3^{(m)}$, so that we would conjecture

\begin{equation}
(L_p(\Omega), \tilde{W}^m_p(\Omega))_{\theta,p} = W^{m\theta}_p(\Omega) \cap L_p(\Omega, d(x)^{-m\theta}) = \tilde{W}^{m\theta}_p(\Omega). \nonumber \end{equation}

However, this is not true. The next section contains statements about the spaces $(L_p(\Omega), \tilde{W}^m_p(\Omega))_{\theta,p}$; it turns out that the interpolation satisfies (11) for certain values of $\theta$, but not for all.
1.4. SOME GENERALISATIONS

In this section we want to suggest in which way the above results can be generalized.

Remark 1. Theorems 1.1, 1.2, 1.3 are also true if the functor \((\cdot, \cdot)_{\theta, r}\) is replaced by a functor \((\cdot, \cdot)_{\theta, q}\), \(1 < q < \infty\). The spaces \((L_p(\Omega), L_p(\Omega), \varphi))_{\theta, q}\) are the so-called Beurling spaces (see J. E. Gilbert [2]). The spaces \((L_p(\Omega), W^m_\varphi(\Omega))_{\theta, q}\) are the usual Besov spaces (see H. Triebel [16] Sect. 4.3.1.1).

This remark can be applied to all the following results. Pay attention to the question whether the spaces on the right hand side in \((\ast)\) are known.

Remark 2. Let \(\Omega\) be an arbitrary domain in \(R^n\). We put in 1.1 and in 1.2:

(i) \(A_0 = L_p(\Omega),\)

\[ A_2 = W_q^t(\Omega), \quad 1 < q < \infty; \quad 0 \leq t < \infty; \quad t - n/q \geq -n/p. \]

Let \(\Omega\) be a bounded domain in \(R^n\) with a \(C^\infty\)-boundary. We put in 1.1, 1.2, 1.3:

(ii) \(A_0 = L_p(\Omega),\)

\[ A_2 = \tilde{W}_q^t(\Omega), \quad 1 < q < \infty; \quad 0 \leq t < \infty; \quad t - n/q \geq -n/q. \]

In (i) and in (ii) the operator \(A\) is the same as in Sections 1.1, 1.2.

In these cases Theorems analogous to Theorems 1.1, 1.2 and 1.3, respectively, are true; the proof is based on the embedding assertions

\[ W_q^t(\Omega) \subset L_p(\Omega), \quad \tilde{W}_q^t(\Omega) \subset L_p(\Omega), \]

which hold due to the assumptions on the domain \(\Omega\) and the parameters \(p, q\) and \(t\) mentioned above.

For the real \(K\)-functor \((\cdot, \cdot)_{\theta, r}\) with

\[ \frac{1}{r} = \frac{1 - \theta}{p} + \frac{\theta}{q}, \]

the spaces on the right hand side of the equation mentioned in Remark 1 are known. We refer to Remark 1 and to H. Triebel [16], Section 4.3.

Remark 3. We obtain an essential generalisation of the above results to other spaces by using the well-known interpolation property (see H. Triebel [16] Ch. 1). The proofs of Theorems 1.1, 1.2 and 1.3 show that the operators \((tA + E)^{-1}, 0 < t < \infty\), acting in \(A_0\) belong to \(\mathcal{L}(\{A_0, A_2\}, \{A_0, A_2\})\). (This set consists of all linear maps from \(A_0 + A_2\) into \(A_0 + A_2\), whose restrictions to \(A_0\) and \(A_2\) are continuous maps in \(A_0\) and \(A_2\), respectively.) For operators belonging to this set the interpolation property implies that their restrictions to \(F(\{A_0, A_2\})\) are bounded.
operators. Thus provided the $A_0$- and $A_2$-conditions are satisfied, we obtain an $F([A_0, A_2])$-condition and

$$ (A_0, F([A_0, A_2]) \cap A_2)_{\theta, q} = (A_0, F([A_0, A_2])_{\theta, q} \cap A_0, A_2)_{\theta, q} $$

holds.

We give some examples: Let $\Omega$ be a bounded domain in $R_\nu$ with a $C^\infty$-boundary or $\Omega = R_\nu$. We denote by $H^s_p(\Omega)$, $1 < p < \infty$, $0 \leq s < \infty$, the usual Lebesgue (Bessel potential or Liouville) spaces and by $B^s_{p,q}(\Omega)$ the usual Besov spaces. We refer to H. Triebel [16] for the definition, the basic properties and the following assertions:

(1) \hspace{1cm} 1 < p < \infty ; \hspace{0.5cm} s = 0, 1, 2, \ldots ; \hspace{0.5cm} H^s_p(\Omega) = W^s_p(\Omega) ;

(2) \hspace{1cm} \left[ H^{s_0}_{p_0}(\Omega), H^{s_1}_{p_1}(\Omega) \right]_{\theta} = H^s_p(\Omega)

\hspace{1cm} \begin{cases}
0 < \theta < 1, & \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \\
1 < p < \infty, & \\
0 < s_0, s_1 < \infty, & s = (1 - \theta) s_0 + \theta s_1 ;
\end{cases}

(3) \hspace{1cm} (H^s_p(\Omega), H^s_p(\Omega))_{\theta, p} = B^s_{p,p}(\Omega) \hspace{0.5cm} 0 < \theta < 1, \hspace{0.5cm} 1 < p < \infty ;

\hspace{1cm} s = (1 - \theta) s_1 + \theta s_2 ;

(4) \hspace{1cm} (W^s_p(\Omega), W^s_p(\Omega))_{\theta, q} = B^s_{p,q}(\Omega)

\hspace{1cm} \begin{cases}
0 < \theta < 1, & 0 \leq s, \hspace{0.5cm} s_1 < \infty, \hspace{0.5cm} s_0 \neq s_1 ; \\
1 < p, \hspace{0.5cm} q < \infty, & \\
s = (1 - \theta) s_0 + \theta s_1 .
\end{cases}

In (2) $[\ldots]_{\theta}$ denotes the complex interpolation functor with parameter $\theta$. If, in addition, the parameters $s_0, s_1$ and $s_2$ fulfill $s_0 - (1/p) \neq \text{integer}$, $s_1 - (1/p) \neq \text{integer}$, $s - (1/p) \neq \text{integer}$, then, replacing $H^s_p(\Omega)$ by $H^s_p(\Omega)$, $W^s_p(\Omega)$ by $W^s_p(\Omega)$ and $B^s_{p,q}(\Omega)$ by $B^s_{p,q}(\Omega)$, similar formulas are true. Thus, we obtain analogous results as in the sections above. We only have to replace $W^m_p(\Omega)$ by $W^m_p(\Omega) = \left[ L^p_p(\Omega), W^m_p(\Omega) \right]_{\sigma/m}$ and by $B^s_{p,q}(\Omega) = (L^p_p(\Omega), W^m_p(\Omega))_{\sigma/m,q}$, respectively. Under the mentioned conditions on the parameters the same holds for $A_2 = H^s_p(\Omega)$ or $A_2 = B^s_{p,q}(\Omega)$.

2. MULTIPLICATION OPERATORS IN WEIGHTED SPACES

In this section we get results of the type (\#) in the case that the Banach spaces $A_0$ and $A_2$ as well as the space $A_1$ are weighted Sobolev spaces. We consider two types of weighted spaces; the spaces $W^m_p(\Omega, \sigma)$ (the weight-function is the same for all the orders of differentiation) and the spaces $W^m_p(\Omega; \sigma^a; \sigma^b)$ (the weight-function depends on the order of differentiation). In both cases we get results of the type (\#) for some classes of weight-functions (described by growth-conditions as in Sections 1.1, 1.2
and 1.3). We state the theorems without proofs, because these can be obtained by obvious modifications of the proofs of Theorems 1.1, 1.2 and 1.3. Nevertheless, we give the necessary definitions.

2.1. MULTIPLICATION OPERATORS IN THE SPACES $W_p^m(\Omega, \sigma)$

Let $\Omega$ be a domain in $R_n$. Let $1 < p < \infty$ and let $m$ be an integer, $m \geq 0$. Then we denote by $W_p^m(\Omega, \sigma)$ the space consisting of all measurable complex-valued functions $u(x)$, which have measurable derivatives up to the $m$-th order belonging to $L_p(\Omega, \sigma)$. The norm is given by

$$\|u\|_{W_p^m(\Omega, \sigma)} = \left( \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L_p(\Omega, \sigma)}^p \right)^{1/p},$$

$\hat{W}_p^m(\Omega, \sigma)$ denotes the closure of $C_0^\infty(\Omega)$ in $W_p^m(\Omega, \sigma)$.

(2) We will consider only such spaces $W_q^m(\Omega, \sigma)$ and $\hat{W}_p^m(\Omega, \sigma)$, for which the interpolations $(L_p(\Omega, \sigma), W_p^m(\Omega, \sigma))_{\theta, p}$ and $(L_p(\Omega, \sigma), \hat{W}_p^m(\Omega, \sigma))_{\theta, p}$, respectively, are well-known. This is true, for instance, for the weighted spaces considered by H. Triebel in [16], Ch. 3. In what follows we assume that the domains considered satisfy the same conditions as those introduced in the reference just mentioned. Now we specify the spaces $A_0$, $A_1$ and $A_2$ of Grisvard's proposition. Let $1 < p < \infty$ and let $m$ be an integer, $m \geq 0$. Let $\varphi(x)$ be a weight-function.

(3) $A_0 = L_p(\Omega, \sigma), \quad A_2 = W_p^m(\Omega, \sigma),$

$$A_1 = [D(A), \|\cdot\|_{A}] = L_p(\Omega, \sigma \varphi).$$

(3') $A_0 = L_p(\Omega, \sigma), \quad A_2 = \hat{W}_p^m(\Omega, \sigma); \quad$ the operator $A$ is the same as in (3).

**Theorem 1.** Let $\Omega$ be a domain described above. The Banach spaces $A_0$, $A_1$ and $A_2$ are given by (3) or (3'); we assume (2). For weight-functions $\varphi(x)$ belonging to the class $K_1^{(m)}$ (cf. Sect. 1.1) Grisvard's conditions are fulfilled. Thus,

(4) $$\left( L_p(\Omega, \sigma), L_p(\Omega, \sigma \varphi) \cap W_p^m(\Omega, \sigma) \right)_{\theta, p} =$$

$$= \left( L_p(\Omega, \sigma \cdot \varphi^0) \cap (L_p(\Omega, \sigma), W_p^m(\Omega, \sigma))_{\theta, p} \right)$$

or

(5) $$\left( L_p(\Omega, \sigma), L_p(\Omega, \sigma \varphi) \cap \hat{W}_p^m(\Omega, \sigma) \right)_{\theta, p} =$$

$$= L_p(\Omega, \sigma \cdot \varphi^0) \cap (L_p(\Omega, \sigma), \hat{W}_p^m(\Omega, \sigma))_{\theta, p},$$

respectively.

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We assume that \( B^s_{pp}(\Omega, \sigma) = (L_p(\Omega, \sigma), W^m_p(\Omega, \sigma))_{s/m, p} \) with \( s = (1 - \theta) m \) and that the spaces \( B^s_{pp}(\Omega, \sigma) \) satisfy the embedding assertion

\[
\text{for } 0 \leq t \leq s \quad \text{and} \quad \infty > q \geq p > 1, \quad B^s_{pp}(\Omega, \sigma) \subset B^t_{q(\Omega, \sigma)}; \\
\frac{s - n}{p} \geq \frac{t - n}{q}.
\]

Examples of such weighted spaces are given in the next section. (6) is true, for instance, if \( \sigma(x) \) belongs to the class \( K^{(m)}_x \), \( s \leq m \).

**Theorem 2.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) described above. The Banach spaces \( A_0, A_1 \) and \( A_2 \) are given by (3); we assume (2) and (6). For weight-functions \( \varphi(x) \) belonging to \( K^{(m)}_x \) (cf. Sect. 1.2), (4) holds.

### 2.2. Multiplication Operators in the Spaces \( W^m_p(\Omega; \varphi^\mu; \varphi^\nu) \)

Let \( \Omega \) be a domain in \( \mathbb{R}^n \). If \( 1 \leq p < \infty \), then by \( L^{\text{loc}}_p(\Omega) \) we mean the spaces, consisting of all complex-valued locally \( p \)-integrable functions \( u(x) \). Outside \( \Omega \) these functions are continued by zero. \( C^\infty(\Omega) \) denotes the set of all complex-valued infinitely differentiable functions in \( \Omega \). By \( \varphi(x) \) we denote a weight-function mapping \( \Omega \) into \( R_1 \), which satisfies

\[
|\nabla \varphi(x)| \leq c \varphi^2(x)
\]

and

(2) for each positive number \( K \) there exist numbers \( \varepsilon_K > 0 \) and \( r_K > 0 \) such that \( \varphi(x) > K \) provided \( d(x) \leq \varepsilon_K \) or \( |x| \geq r_K, x \in \Omega \).

We introduce the space \( W^s_p(\Omega; \varphi^\mu; \varphi^\nu) \) and refer to H. Triebel [16], Ch. 3 for details.

**Definition.** Let \( \Omega \) be an arbitrary domain in \( \mathbb{R}^n \) and \( \varphi(x) \) a weight-function in the sense of (1) and (2). Let \( p, s, \mu \) and \( \nu \) be real numbers with \( 1 \leq p < \infty, s \geq 0 \) and \( \nu \geq \mu + s \). Then we put

\[
W^s_p(\Omega; \varphi^\mu; \varphi^\nu) = \left\{ u : u \in L^{\text{loc}}_p(\Omega); \|u\|_{W^s_p(\Omega; \varphi^\mu; \varphi^\nu)} = \right\}
\]

\[
= \left[ \int_\Omega \left( \sum_{|x| = m} \varphi^\mu(x) |D^s u(x)|^p + \varphi^\nu(x) |u(x)|^p \right) dx \right]^{1/p} < \infty
\]

for \( s = 0, 1, 2, \ldots; \) for \( s = 0 \) let \( \mu = \nu; \) \( W^0_p(\Omega; \varphi^\mu; \varphi^\nu) = L_p(\Omega, \varphi^\mu) \)
and

\[ W^{s}_{p}(\Omega; q^{\mu}; q^{\nu}) = \left\{ u : u \in L^{\text{loc}}_{p}(\Omega) ; \| u \|_{W^{s}_{p}(\Omega; q^{\mu}; q^{\nu})} = \right\} \]

\[ = \left[ \int_{\Omega \times \Omega} \sum_{|s| = |x|} |q^{\mu}(x) D^{s}u(x) - q^{\mu}(y) D^{s}u(y)|^{p} \frac{dxdy}{|x - y|^{n + |s|p}} \right]^{1/p} < \infty \]

for \( 0 < s \neq \text{integer}, s = \lfloor s \rfloor + \{ s \}, \lfloor s \rfloor \text{ integer}, 0 < \{ s \} < 1. \)

We introduce an equivalent norm in \( W^{s}_{p}(\Omega; q^{\mu}; q^{\nu}) \): Let the assumptions of Definition 1 be fulfilled. Let further \( 0 \leq t \leq s \) and

\[ \chi_{t} = \mu \frac{t}{s} + v \frac{s - t}{s} = \mu + (v - \mu) \frac{s - t}{s}. \]

Then

\[ (\| u \|_{W^{s}_{p}(\Omega; q^{\mu}; q^{\nu})}^{p})^{p} = (\| u \|_{W^{s}_{p}(\Omega; q^{\mu}; q^{\nu})}^{p})^{p} + \sum_{|s| = |x|} \int_{\Omega} q^{\nu}(x) |D^{s}u(x)|^{p} \frac{dx}{|x|^{n + |s|p}} \]

defines an equivalent norm in \( W^{s}_{p}(\Omega; q^{\mu}; q^{\nu}) \).

**Definition 2.** Let \( \Omega \) be a bounded domain in \( R_{n} \). Let \( q(x) \) be a weight-function in the sense of (1) and (2) for which, in addition, the identity \( q^{-1}(x) = d(x) \) holds near to \( \partial \Omega \). Then we set for \( 1 < p < \infty, s \geq 0 \) and \( v < \mu + s: \)

\[ W^{s}_{p}(\Omega; q^{\mu}; q^{\nu}) = \left\{ u : u \in L^{\text{loc}}_{p}(\Omega) ; \| u \|_{W^{s}_{p}(\Omega; q^{\mu}; q^{\nu})} = \right\} \]

\[ = \left[ \int_{\Omega} \left( \sum_{|s| = |x|} q^{\mu}(x) |D^{s}u(x)|^{p} + q^{\nu}(x) |u(x)|^{p} \right) dx \right]^{1/p} < \infty \]

if \( s \) is an integer. For \( s = 0 \) we put \( \mu = v \) : \( W^{0}_{p}(\Omega; q^{\mu}; q^{\nu}) = L^{p}_{p}(\Omega; q^{\mu}) \)

while

\[ W^{s}_{p}(\Omega; q^{\mu}; q^{\nu}) = \left\{ u : u \in L^{\text{loc}}_{p}(\Omega) ; \| u \|_{W^{s}_{p}(\Omega; q^{\mu}; q^{\nu})} = \right\} \]

\[ = \left[ \int_{\Omega \times \Omega} \sum_{|s| = |x|} |q^{\mu}(x) D^{s}u(x) - q^{\mu}(y) D^{s}u(y)|^{p} \frac{dxdy}{|x - y|^{n + |s|p}} \right]^{1/p} + \| u \|_{W^{s}_{p}(\Omega; q^{\mu}; q^{\nu})} < \infty \]

for \( s = \lfloor s \rfloor + \{ s \}, \) where \( \lfloor s \rfloor \) is an integer and \( 0 < \{ s \} < 1. \)

\( \hat{W}^{s}_{p}(\Omega; q^{\mu}; q^{\nu}) \) denotes the closure of \( C^{\infty}_{c}(\Omega) \) in \( \hat{W}^{s}_{p}(\Omega; q^{\mu}; q^{\nu}) \) and by \( \hat{W}^{s}_{p}(\Omega; q^{\mu}; q^{\nu}) \) we mean the closure of \( C^{\infty}_{c}(\Omega) \) in \( \hat{W}^{s}_{p}(\Omega; q^{\mu}; q^{\nu}). \)

We note that (5) holds as well. Now we state embedding assertions analogous to the statements 1.2 (3) and 2.1 (6).
Proposition. (i) Let \( \Omega \) be an arbitrary domain and \( \varphi(x) \) a weight-function in the sense of (1) and (2). Let \( t, q, p, s, \mu \) and \( \nu \) be real numbers with

\[
s \geq t \geq 0, \quad \infty > q \geq p > 1, \quad s - \frac{n}{p} = t - \frac{n}{q}, \quad \nu > \mu + s.
\]

Then

\[
(8) \quad W_p^\mu(\Omega; \varphi^\mu; \varphi^\nu) \subseteq W_q^\nu(\Omega; \varphi^\nu; \varphi^\nu); \quad \text{with} \quad \chi = \mu \quad \text{and} \quad \tau = \mu \left( \frac{s - t}{s} \right) + \nu t.
\]

(ii) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with a \( C^\infty \)-boundary and let \( \varphi(x) \) be a weight-function in the sense of (1) and (2) for which, in addition \( \varphi^{-1}(x) = d(x) \) holds near to \( \partial \Omega \). Let \( t, q, p, s, \mu \) and \( \nu \) be real number with

\[
s \geq t \geq 0, \quad \infty > q \geq p > 1, \quad s - \frac{n}{p} > t - \frac{n}{q}.
\]

Then

\[
(9) \quad \hat{W}_p^\mu(\Omega; \varphi^\mu; \varphi^\nu) \subseteq \hat{W}_q^\nu(\Omega; \varphi^\nu; \varphi^\nu) \quad \text{with} \quad \chi = \mu \quad \text{and} \quad \tau < \chi + t.
\]

(iii) Let the assumptions of (ii) be satisfied. Further, let

\[
\{s\} = \frac{1}{p}, \quad \mu + s = \frac{1}{p} + k \quad \text{with} \quad k = 0, 1, \ldots, \lfloor s \rfloor - 1,
\]

\[
\{t\} = \frac{1}{p}, \quad \chi + t = \frac{1}{p} + l \quad \text{with} \quad l = 0, 1, \ldots, \lfloor t \rfloor - 1.
\]

Then (9) holds for

\[
s - \frac{n}{p} \geq t - \frac{n}{q}.
\]

After these preliminaries we formulate

Theorem. (a) Let \( \Omega \) be an arbitrary domain in \( \mathbb{R}^n \) and \( \varphi(x) \) a weight-function in the sense of (1) and (2). If \( 1 < p < \infty, \infty > s \geq \mu, \nu > \mu + s \) and if \( \varphi(x) \) is a weight-function, we put:

\[
A_0 = L_p(\Omega, \varphi^\nu), \quad A_2 = W_p^\nu(\Omega; \varphi^\nu; \varphi^\nu), \quad (Au)(x) = \varphi(x)u(x),
\]

\[
D(A) = \{u : u \in L_p(\Omega, \varphi^\nu); \quad \varphi u \in L_p(\Omega, \varphi^\nu), \quad \varphi^s u \in L_p(\Omega, \varphi^\nu), \quad \varphi^s \varphi u \in L_p(\Omega, \varphi^\nu) \},
\]

\[
A_1 = [D(A), \|\cdot\|_{s_0}] = L_p(\Omega, \varphi^\nu \varphi).
\]

For functions \( \varphi(x) \) belonging to a class \( K^{(m)}_1 \) with \( m \geq s \), Grisvard's conditions are satisfied and

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\[(10)\quad (L_p(\Omega, \varphi^s), L_p(\Omega, \varphi^\alpha) \cap W^s_p(\Omega; \varphi^{\omega}; \varphi^s))_{\theta, p} =
= L_p(\Omega, \varphi^\alpha \varphi^\theta) \cap W^s_p(\Omega; \varphi^{\omega}; \varphi^s)
\]
holds with \(\tau = \theta(\mu - v) + v\).

(b) Let \(\Omega \subset \mathbb{R}_n\) be an arbitrary domain and let \(\varrho(x)\) be a weight-function in the sense of (1) and (2). Let the Banach spaces \(A_0, A_1\) and \(A_2\) be the same as in (a). For weight-functions \(\varphi(x)\) belonging to a class \(K^{(m)}_2\) which \(m \geq s\), (10) holds.

(c) Let \(\Omega\) be a bounded domain in \(\mathbb{R}_n\) with a \(C^\infty\)-boundary. Let \(\varrho(x)\) be the same weight-function as in Definition 2 and let \(\varphi(x)\) be a weight-function. If \(p, s, \mu\) and \(v\) are real numbers with \(1 < p < \infty, 0 \leq s < \infty, v < \mu + s, \{s\} = 1/p, \mu + s = (1/p) + k\) for \(k = 0, 1, \ldots, [s] - 1\), we put:

\[A_0 = L_p(\Omega, \varphi^s), \quad A_2 = \tilde{W}^s_p(\Omega; \varphi^{\omega}; \varphi^s), \quad (Au)(x) = \varphi(x)u(x),\]

\[A_1 = L_p(\Omega, \varphi^\alpha).\]

For weight-functions \(\varphi(x)\) belonging to a class \(K^{(m)}_2\) with \(m \geq s\) Grisvard's conditions are satisfied and

\[(11)\quad (L_p(\Omega, \varphi^s), L_p(\Omega, \varphi^\alpha) \cap \tilde{W}^s_p(\Omega; \varphi^{\omega}; \varphi^s))_{\theta, p} =
= L_p(\Omega, \varphi^\alpha \varphi^\theta) \cap W^s_p(\Omega; \varphi^{\omega}; \varphi^{s+\omega_p}).\]

Remark 1. In the case that the conditions of (c) are satisfied we can replace in (a) and (b) the spaces \(W^s_p(\Omega; \varphi^{\omega}; \varphi^s)\) by \(\tilde{W}^s_p(\Omega; \varphi^{\omega}; \varphi^s)\), so that an analogous formula to formula (10) holds.

Remark 2. Using the same argument as in Section 1.1 we put the statements (a) and (b) together obtaining the following

**Corollary.** Let \(\Omega\) be an arbitrary domain in \(\mathbb{R}_n\) and \(\varrho(x)\) a weightfunction in the sense of (1) and (2). Let \(A_0, A_1\) and \(A_2\) be the same Banach spaces as in Theorem (a). Let \(\varphi(x)\) be a weight-function satisfying

\[(1)\quad \frac{1}{\varphi(x)} \in L^{\infty}(\Omega),\]

\[(2)\quad \text{for all } 1 \leq |x| \leq m \text{ there exist numbers } q_x, n^1|x| \leq q_x \leq \infty, \text{ such that }\]

\(\frac{\partial^\alpha \varphi}{\varphi} \in L_{q_x}(\Omega)\).

Then (10) holds.
3. THE INTERPOLATION $(W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1) \cap W^r_p(\Omega, \sigma_0))_{\theta,p}$, $r \geq m$

In this section we describe the spaces

$$ (W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1) \cap W^r_p(\Omega, \sigma_0))_{\theta,p}, \quad r, m \text{ integers}, $$

$$ r \geq m \geq 0, $$

for certain classes of pairs $(\sigma_0(x), \sigma_1(x))$ of weight-functions. First we present some preliminaries.

3.1. THE SPACES $(W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1))_{\theta,p}$

Let $\Omega$ be an arbitrary domain in $\mathbb{R}_n$ and let $\sigma_0(x)$ and $\sigma_1(x)$ be weight-functions. For $m = 0, 1, 2, \ldots$ and $1 < p < \infty$ the spaces $W^m_p(\Omega, \sigma)$ are defined as in Section 2.1. Favini showed in [1] that the spaces $(W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1))_{\theta,p}$ satisfy the embedding assertion

$$ (W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1))_{\theta,p} \subseteq W^m_p(\Omega, \sigma_\theta), \quad 0 < \theta < 1, $$

$$ 1 < p < \infty, \quad \sigma_\theta = \sigma_0^{1-\theta}\sigma_1^\theta. $$

We will show that under certain growth conditions on the pair $(\sigma_0(x), \sigma_1(x))$ of weight-functions the inverted embedding holds as well. To this aim we need a proposition from the interpolation theory of positive operators; for the proposition we refer to H. Triebel [16], Ch. 1.

**Definition.** Let $A$ be a closed operator acting in $A$ with a dense domain of definition $D(A)$. The operator $A$ is said to be positive, if $(-\infty, 0]$ is contained in the resolvent set of $A$ and if there exists a number $c$, $c \geq 0$, such that

$$ \|(A - tE)^{-1}\| \leq \frac{c}{1 + |t|}; \quad t \in (-\infty, 0]. $$

An equivalent inequality to (2) is $\|(tA + E)^{-1}\| \leq c/(1 + t), \quad t \in [0, \infty)$.)

**Proposition.** Let $A$ be a positive operator acting in $A$. Let $m$ be a natural number, $0 < \theta < 1$, and let $1 < p < \infty$. Then

$$ (A, D(A^m))_{\theta,p} = \{a : a \in A \|a\|^* = \|\text{im}[A(A + tE)^{-1}]^m a\|_{L^p(A)} < \infty\} $$

is valid, when putting

$$ \|v(t)\|_{L^p(A)} = \left(\int_0^\infty \|v(t)\|^p_A \frac{dt}{t}\right)^{1/p} $$

with the usual modifications for $p = \infty$. 357
After this preliminaries we present

**Theorem 1.** Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$ and let $m$ be a natural number. We assume that $1 < p < \infty$ and that $\sigma_0(x)$, $\sigma_1(x)$ are weight-functions satisfying

\begin{equation}
D^x \left( \frac{\sigma_1}{\sigma_0} \right) \leq c \frac{\sigma_1}{\sigma_0}, \quad 0 < |x| \leq m,
\end{equation}

and

\begin{equation}
\frac{\sigma_0}{\sigma_1} \in L^\infty(\Omega).
\end{equation}

Then the operator $(Au)(x) = (\sigma_1(x)/\sigma_0(x)) u(x)$, $D(\Lambda) = \{ u : u \in W^m_p(\Omega, \sigma_0), (\sigma_1/\sigma_0) u \in W^m_p(\Omega, \sigma_0) \}$, is a positive operator acting in $W^m_p(\Omega, \sigma_0)$. Further, $D(\Lambda) = W^m_p(\Omega, \sigma_1)$ up to equivalent norms and for $0 < \theta < 1$, $1 < p < \infty$, the following identity holds:

\begin{equation}
(W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1))_\theta,p = W^m_p(\Omega, \sigma_\theta), \quad \text{where} \quad \sigma_\theta = \sigma_0^{1-\theta} \sigma_1^\theta.
\end{equation}

**Proof. First step:** We show that $D(\Lambda) = W^m_p(\Omega, \sigma_1)$. The inclusion $D(\Lambda) \subseteq W^m_p(\Omega, \sigma_1)$: Let $u(x) \in D(\Lambda) = \{ u : u \in W^m_p(\Omega, \sigma_0), (\sigma_1/\sigma_0) u \in W^m_p(\Omega, \sigma_0) \}$, which is equipped with the norm

\[ \| u \|_{D(\Lambda)} = \| u \|_{W^m_p(\Omega, \sigma_0)} + \left[ \sum_{0 \leq |x| \leq m} \| D_x \left( \frac{\sigma_1}{\sigma_0} \right)^p \right]^{1/p}. \]

Then

\begin{equation}
D_x \left( \frac{\sigma_1}{\sigma_0} u \right) = \frac{\sigma_1}{\sigma_0} D^x u + \sum_{|x| \geq 1} c_j D^x \left( \frac{\sigma_1}{\sigma_0} \right) D^{x-j} u
\end{equation}

is valid. Consequently,

\[ \| u \|_{W^m_p(\Omega, \sigma_1)} = \sum_{0 \leq |x| \leq m} \| D^x u \|_{L^p(\Omega, \sigma_1)} = \sum_{0 \leq |x| \leq m} \| D^x u \|_{L^p(\Omega, \sigma_1)} + \sum_{|x| = m} \frac{\sigma_1}{\sigma_0} \| D^x u \|_{L^p(\Omega, \sigma_0)} \leq \sum_{0 \leq |x| \leq m-1} \| D^x u \|_{L^p(\Omega, \sigma_1)} + \sum_{|x| = m} \| D^x \left( \frac{\sigma_1}{\sigma_0} u \right) \|_{L^p(\Omega, \sigma_0)} \]

\[ \leq \sum_{|x| = m} \| D^x \left( \frac{\sigma_1}{\sigma_0} u \right) \|_{L^p(\Omega, \sigma_0)} + c \sum_{|x| = m} \| D^x \left( \frac{\sigma_1}{\sigma_0} u \right) \|_{L^p(\Omega, \sigma_0)} \]

By iteration we get $\| u \|_{W^m_p(\Omega, \sigma_1)} \leq c \| u \|_{D(\Lambda)}$, where $c$ is a positive constant.
The inclusion \( W^m_p(\Omega, \sigma_1) \subseteq D(A) \):

We have to show

\[
(9) \quad \left\| D^x \left( \frac{\sigma_1}{\sigma_0} u \right) \right\|_{L^p(\Omega, \sigma_0)}^p \leq c \| u(x) \|_{W^m_p(\Omega, \sigma_1)}^p, \quad 0 \leq |x| \leq m, \quad u \in W^m_p(\Omega, \sigma_1).
\]

Using (8) we have

\[
(10) \quad \left\| D^x \left( \frac{\sigma_1}{\sigma_0} u \right) \right\|_{L^p(\Omega, \sigma_0)}^p \leq c \left\| D^x u \right\|_{L^p(\Omega, \sigma_1)}^p + 
\]

\[
+ c \sum_{|\gamma| \geq 1} \left\| \frac{D^\gamma \left( \frac{\sigma_1}{\sigma_0} \right)}{\sigma_1^{\gamma} \sigma_0} \right\|_{L^p(\Omega, \sigma_0)}^p \leq c \sum_{0 \leq |\gamma| \leq |\gamma|} \left\| D^\gamma u \right\|_{L^p(\Omega, \sigma_1)}^p = c \left\| u \right\|_{W^m_p(\Omega, \sigma_1)}^p.
\]

Second step: We recall Theorem 1.1. The properties (5) and (6) of \( \sigma_1(x)/\sigma_0(x) \) correspond to the properties 1.1 (1) and 1.1 (2) of a weight-function \( \phi(x) \) belonging to a class \( K^{(m)} \). Thus, the proof of the positivity of the operator \( A \) is obtained by obvious modifications of the proof of Theorem 1.1.

Third step: To show

\[
(W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1))_{0,p} = W^m_p(\Omega, \sigma_0^{1-\theta} \sigma_1^{\theta}).
\]

The inclusion \( (W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1))_{0,p} \subseteq W^m_p(\Omega, \sigma_0^{1-\theta} \sigma_1^{\theta}) \) was proved by A. Favini \([1]\). We show the converse inclusion. Using the above Proposition we have to show that any function \( u(x) \in W^m_p(\Omega, \sigma_0^{1-\theta} \sigma_1^{\theta}) \) satisfies

\[
(11) \quad u \in W^m_p(\Omega, \sigma_0),
\]

\[
(12) \quad \int_0^\infty t^{\rho_p} \left\| A (A + tE)^{-1} u \right\|_{W^m_p(\Omega, \sigma_0)}^p \frac{dt}{t} \leq c \left\| u \right\|_{W^m_p(\Omega, \sigma_0^{1-\theta} \sigma_1^{\theta})}^p.
\]

For an arbitrary function \( u(x) \in W^m_p(\Omega, \sigma_0^{1-\theta} \sigma_1^{\theta}) \) we have

\[
(13) \quad \int_\Omega \left| \sigma_0 D^x u \right|^p dx = \int_\Omega \left| \sigma_0^{1-\theta} \sigma_1^{\theta} D^x u \right|^{\rho_p} \left( \frac{\sigma_0}{\sigma_1} \right)^{\rho_p} \left( \frac{\sigma_0^{1-\theta} \sigma_1^{\theta}}{\sigma_0^{1-\theta} \sigma_1^{\theta}} \right) dx \leq c \left\| D^x u \right\|_{L^p(\Omega, \sigma_0^{1-\theta} \sigma_1^{\theta})}^p,
\]

so that \( u(x) \) belongs to \( W^m_p(\Omega, \sigma_0) \).

To show (12) we write the integral on the left hand side as

\[
\int_0^\infty \sum_{0 \leq |x| \leq m} \left\| D^x \left( \frac{\sigma_1}{\sigma_0} \right) \right\|_{L^p(\Omega, \sigma_0)}^p \frac{dt}{t}.
\]

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We consider only two summands. First summand: \( \alpha = (0, \ldots, 0) = \alpha_0; \)
\[
\int_0^\infty t^{\theta_p} \left| \frac{\sigma_1}{\sigma_0} - \frac{\sigma_1}{\sigma_0 + t} \right|_L^p \frac{dt}{t} = \\
= \int_0^1 t^{\theta_p} \left| \frac{\sigma_1}{\sigma_0} - \frac{\sigma_1}{\sigma_0 + 1} \right|_L^{p} \frac{dt}{t} + \int_1^\infty t^{\theta_p} \left| \frac{\sigma_1}{\sigma_0} - \frac{\sigma_1}{\sigma_0 + 1} \right|_L^{p} \frac{dt}{t} \leq \\
\leq c_1 \left\| u \right\|_{L_p(\Omega, \sigma_0)}^{p} + \int_1^\infty \frac{dt}{t} \left( \frac{\sigma_0}{\sigma_1} \right)^{1-\theta} \left( \frac{\sigma_0}{\sigma_1} \right)^{\theta} \left\| u \right\|_{L_p(\Omega)}^{p} \leq c_2 \left\| u \right\|_{L_p(\Omega, \sigma_0)}^{p},
\]
where \( \varepsilon \) is a suitable real number, \( 0 < \varepsilon < \theta \).

Second summand: \( \alpha = (0, \ldots, 0, 1) = \alpha_1; \)
\[
(14) \quad \int_0^\infty t^{\theta_p} \left| \frac{\partial}{\partial x_n} \left( \frac{\sigma_1}{\sigma_0} - \frac{\sigma_1}{\sigma_0 + t} \right) \right|_L^p \frac{dt}{t} \leq \int_0^\infty t^{\theta_p} \left| \frac{\partial}{\partial x_n} \left( \frac{\sigma_1}{\sigma_0} \right) \right|_L^p + \left| \frac{\partial}{\partial x_n} \left( \frac{\sigma_1}{\sigma_0} \right) \right|_L^p \frac{dt}{t},
\]

With the aid of the above decomposition of \([0, \infty)\) and using the properties (5) and (6) of \( \sigma_i(x)/\sigma_0(x) \), we further estimate (14) and get the desired result. In the same way the summands
\[
\int_0^1 t^{\theta_p} \left( \frac{\sigma_1}{\sigma_0} - \frac{\sigma_1}{\sigma_0 + t} \right) \left| D^x \left( \frac{\sigma_1}{\sigma_0} - \frac{\sigma_1}{\sigma_0 + t} \right) \right|_L^p \frac{dt}{t}
\]
with \( \alpha \neq \alpha_0, \alpha_1 \) can be estimated, so that the theorem is proved.
Remark 1. From the second step in the proof we deduce an embedding assertion for weight Sobolev spaces:

**Corollary.** Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. Let $m$ be an integer and $\sigma(x)$ a weight-function with

$$ \frac{D^s\sigma}{\sigma} \in L^\infty(\Omega) \quad \text{for} \quad 0 < |x| \leq m.$$  

Then

$$ W^s_p(\Omega, \sigma) \subseteq W^t_q(\Omega, \sigma) $$

holds provided $\infty > q \geq p > 1$, $0 \leq t \leq s \leq m < \infty$ and

$$ \frac{s - n}{p} \geq \frac{t - n}{q}. $$

**Proof.** Let us assume that $s$ is an integer. Then $u \in W^s_p(\Omega, \sigma)$ implies that $\sigma u \in W^s_p(\Omega)$. The embedding assertion 1.2 (3) for unweighted Sobolev spaces implies that $\sigma u \in W^t_q(\Omega, \sigma)$, so that $u \in W^t_q(\Omega, \sigma)$. By means of interpolation we extend the result to real values $s, s \neq \text{integer}.$

Remark 2 (Examples). All weight-functions $\sigma_0(x)$ and $\sigma_1(x)$ (in the sense of Section 1.1) whose quotient $\sigma_1(x)/\sigma_0(x)$ belongs to a class $K_1^{m}$ are examples of pairs $(\sigma_0(x), \sigma_1(x))$ of weight-functions in the sense of Theorem 1. So we refer to Remark 1.1.2.

After these preliminaries we formulate

**Theorem 2.** Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$ and let $\sigma_0(x)$ and $\sigma_1(x)$ be weight-functions satisfying (5) and (6). For $1 < p < \infty$, natural numbers $m$ and $r, r > m$, we put $A_0 = W^m_p(\Omega, \sigma_0)$, $A_2 = W^r_p(\Omega, \sigma_0)$, 

$$ (Au)(x) = \frac{\sigma_1(x)}{\sigma_0(x)} u(x), \quad D(A) = \left\{ u : u \in W^m_p(\Omega, \sigma_0), \frac{\sigma_1}{\sigma_0} u \in W^r_p(\Omega, \sigma_0) \right\}. $$

Then

$$ A_1 = [D(A), \| \cdot \|_A] = W^m_p(\Omega, \sigma_1), $$

Grisvard's conditions are satisfied and

$$ (W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1) \cap W^r_p(\Omega, \sigma_0))_{\theta, p} = $$

$$ = W^m_p(\Omega, \sigma_1^{\frac{1}{\theta} \sigma_0^\theta}) \cap (W^m_p(\Omega, \sigma_0), W^r_p(\Omega, \sigma_0))_{\theta, p}. $$

**Proof.** (15) follows immediately from Theorem 1. The $A_0$-condition and the $A_2$-condition can be proved in the same way as the positivity of the operator $A$ in Theorem 1. Hence
\[(W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1) \cap W^r_p(\Omega, \sigma_0))_{\theta, p} = \]
\[= (W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_1))_{\theta, p} \cap (W^m_p(\Omega, \sigma_0), W^m_p(\Omega, \sigma_0))_{\theta, p}. \]

The first space on the right hand side of (17) is described in Theorem 1, for the other we refer to H. Triebel [16], Sec. 3.3.

4. EXAMPLES AND COUNTER-EXAMPLES OF THE NON-COMMUTATIVE INTERPOLATION

H. Triebel in [15] gave examples and counter-examples to (*), obtained independently of all the general results mentioned. We extend this result to the spaces introduced by P. Grisvard in [5] and [6], but we use the version of H. Triebel [16], Ch. 4. The spaces under consideration are spaces between \(W^m_p(\Omega)\) and \(\tilde{W}^m_p(\Omega)\) and are determined by differential operators. They are of interest in the theory of differential operators.

Next we give without proof some statements we use throughout this section. We refer to P. Grisvard [5, 6] and to H. Triebel [16], for details and for proofs.

Let \(\Omega\) be a bounded domain in \(\mathbb{R}_n\) with a \(C^\infty\)-boundary. Let \(d(x)\) be the distance from \(x \in \Omega\) to the boundary \(\partial \Omega\). Then we denote by \(L_{p,v}(\Omega) = L_{p,v}\) the space

\[(1) \quad L_{p,v}(\Omega) = \{ u : \|u\|_{L_{p,v}} = \left( \int_\Omega d(x)^p |u(x)|^p \, dx \right)^{1/p} < \infty \} . \]

We define spaces \(W^{m(1)}_p(\Omega)\), characterized by boundary conditions.

**Definition.** Let \(\Omega\) be a bounded domain in \(\mathbb{R}_n\) with a \(C^\infty\)-boundary. Further, let \(1 < p < \infty, 0 < \theta < 1\); let \(m\) be an integer and \(l = 0, 1, 2, \ldots\).

(a) Let us assume that \(m\theta - 1/p = j\) for each number \(j; j = 0, 1, \ldots, l\). Then

\[(3) \quad (L_p(\Omega), W^{m(1)}_p(\Omega))_{\theta, p} = W^{m\theta(1)}_p(\Omega) . \]

(b) If there is \(k \in \{0, \ldots, l\}\), such that \(k = m\theta - 1/p\), then

\[(4) \quad (L_p(\Omega), W^{m(1)}_p(\Omega))_{\theta, p} = \left\{ u : u \in W^{m\theta(1)}_p(\Omega) \int_\Omega d(x)^{-1} |D^\alpha u(x)|^p \, dx < \infty \quad \forall x, \ |x| = k \right\} . \]

**Remark 1.** We give another formulation of the statements (a) and (b):

(a) (i) for \(l = 0, 1, 2, \ldots\) and \(\theta \in \left(0, \frac{1}{pm}\right)\) we have

\[(5) \quad (L_p(\Omega), W^{m(1)}_p(\Omega))_{\theta, p} = W^{m\theta(1)}_p(\Omega) ; \]

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(ii) for \( l < m - 1 \) and \( \theta \in \left( \frac{1 + rp}{mp}, \frac{1 + (r + 1)p}{mp} \right), \ r = 0, \ldots, l - 1, \)
or
\[ \theta \in \left( \frac{1 + rp}{mp}, 1 \right), \ r = l, \] we have
\[ (L_p(\Omega), W^{m(1)}_p(\Omega))_{\theta,p} = \{ u : u \in W^{m\theta}_p(\Omega), D^s u(x)|_{\partial\Omega} = 0, \ |x| \leq r \}; \]

(iii) if \( l \geq m - 1 \) and \( \theta \in \left( \frac{1 + rp}{mp}, \min \left( 1, \frac{1 + (r + 1)p}{mp} \right) \right), \)
\[ r = 0, \ldots, m - 1; \] then (6) again holds;

(b) for \( \theta = \frac{1 + rp}{mp}, \ r = 0, \ldots, \min (l, m - 1), \) we have
\[ (L_p(\Omega), W^{m(1)}_p(\Omega))_{\theta,p} = \{ u : u \in W^{m\theta}_p(\Omega), D^s u(x)|_{\partial\Omega} = 0, \ |x| < r, \ D^s u(x) \in L_{p,-1/p}, \ |x| = r \}. \]

Further, we need two well-known assertions:

(8) For \( 1 < p < \infty, 0 < \theta < 1, \) \( \nu \) real, the identity \( (L_p(\Omega), L_{\nu,v})_{\theta,p} = L_{p,\nu v}^{\theta} \) holds.

(9) For \( 1 < p < \infty, 0 < \theta < 1, \) \( m = 0, 1, \ldots, (L_p(\Omega), W^{m_0}_p(\Omega))_{\theta,p} = W^{\theta}_p(\Omega) \) holds.

We also need the following

**Lemma.** Let \( \Omega \subset R^*_n \) be a bounded domain with a \( C^\infty \)-boundary.

(a) Let \( 1 < p < \infty, \) let \( m \) be a natural number and \( l \) an integer, \( 0 \leq l \leq m - 1. \)
Then \( W^{m(1)}_p(\Omega) = W^{m\nu}_p(\Omega) \cap L_{\nu,v} \) holds provided \( \nu \) satisfies \( -m \leq \nu, -(l + 1) - 1/p < \nu \leq -l - 1/p. \)

(b) For \( 1/p < s \leq 1 \) and \( -s \leq \nu \leq -1/p \) we have
\[ W^{m(\nu)}(\Omega) = \bar{W}^{m\nu}_p(\Omega) = W^s_p(\Omega) \cap L_{\nu,v}, \]
for \( 0 \leq s < 1/p \) we have
\[ W^s_p(\Omega) = W_p^{s(0)} = \bar{W}^s_p = W^s_p \cap L_{\nu,v} \] with \( -s \leq \nu. \)

Remark 2. Combining the statements (a) and (b) of Lemma we conclude: Let \( 1 < p < \infty, \) let \( s \) be a positive number, \( s = \lfloor s \rfloor + \{ s \}, \lfloor s \rfloor = \text{integer and } 0 \leq \lfloor s \rfloor \leq \{ s \} < 1, \) then for \( l = 0, \ldots, \lfloor s \rfloor - 1 \) the identity \( W^{m(1)}_p(\Omega) = W^{m_0}_p(\Omega) \cap L_{\nu,v} \) holds provided \( \nu \) satisfies
\[ -(l + 1) - \frac{1}{p} < \nu \leq -l - \frac{1}{p}, \ -s \leq \nu. \]
If \( s = \lfloor s \rfloor + \{s\} \) with \( \{s\} > 1/p \), then \( W^{(s)}_{p}(\Omega) = W^{(s)}_{p}(\Omega) \cap L_{p,v} \) for \( v \) satisfying
\[-s \leq v \leq -\lfloor s \rfloor \frac{1}{p}.

Remark 3. The interpolation couples \((A_0, A_1)\) and \((A_0, A_2)\) always satisfy
\((A_0, A_1 \cap A_2)_{p} \subseteq (A_0, A_1)_{p} \cap (A_0, A_2)_{p}\)
because \(\max(K(t, a; A_0, A_1), K(t, a; A_0, A_2)) \leq c K(t, a; A_0, A_1 \cap A_2), a \in A_0 + (A_1 \cap A_2),\)
where \(K(t, a; A_0, A_1)\) is the K-functional of the real K-method:
\[
K(t, a; A_0, A_1) = \inf_{a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1.
\]

Remark 4. For \(0 < s \leq 1\) and \(\max(-s, -1/p)\) we have
\[W^{(s)}_{p}(\Omega) \subseteq L_{p,v}.
\]

After these preliminaries we can formulate

Theorem. Let \(\Omega\) be a bounded domain in \(R^n\) with a \(C^\infty\)-boundary, \(d(x)\) denotes the distance from \(x \in \Omega\) to \(\partial \Omega\). Let \(1 < p < \infty\) and let \(m\) be a natural number.

(a) Let \(v\) be a real number satisfying either \(v \leq -m\) or \(v > -1/p\). Then
\[\tag{10}
(L_p(\Omega), W^{m}_{p}(\Omega) \cap L_{p,v})_{\theta,p} = (L_p(\Omega), W^{m}_{p}(\Omega))_{\theta,p} \cap (L_p(\Omega), L_{p,v})_{\theta,p}
\]
provided \(0 < \theta < 1\).

(b) Let \(\max(-(l + 1) - (1/p), -m) < v \leq -l - (1/p), l \in \{0, \ldots, m - 1\}\).

(i) In addition, let there exist numbers \(k, k \in \{0, \ldots, l - 1\}\), with
\[
v < \frac{-(1 + kp)mp}{1 + (k + 1)p};
\]
let \(k_0\) denote the maximum of these numbers. Then (10) holds for
\[
\theta \in \left(0, \frac{1}{mp}\right) \cup \left(-\frac{1}{pv}, \frac{1 + p}{mp}\right) \cup \left[-\frac{1}{pv}, \frac{1 + 2p}{mp}\right) \cup \ldots
\]
\[
\ldots \cup \left[-\frac{1}{pv}, \frac{1 + (k_0 + 1)p}{mp}\right) \cup \left[-\frac{1}{pv}, 1\right) = J_1.
\]

(ii) If there do not exist numbers \(k\) described in (i), then (10) holds for
\[
\theta \in \left[0, \frac{1}{mp}\right) \cup \left[-\frac{1}{pv}, 1\right) = J_2.
\]

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(c) Under the same conditions on \( v \) as in (b) (i) or in (b) (ii), (10) does not hold for \( \theta \in (0, 1) \setminus J_1 \) or \( \theta \in (0, 1) \setminus J_2 \), respectively.

(a) and (b) give examples of non-commutative interpolation while (c) gives a counter-example of non-commutative interpolation.

Proof. First step: for \( v \leq m \) we refer to H. Triebel [7], Sect. 3.4.2. In virtue of (8), (9) and Remarks 3 and 4, formula (10) is immediately clear for \( v > -1/p > -1 \).

Second step: We prove (b): Let

\[
\max \left( -(l + 1) - \frac{1}{p}, -m \right) < v \leq -l - \frac{1}{p}, \ l \in \{0, ..., m - 1\}.
\]

Using (5), (8) and (9) and Remarks 3 and 4, we find that formula (10) holds for \( 0 < \theta < 1/mp \) not only for (b) (i) but for (b) (ii) as well. If \(-1 - lp)/vp < \theta < 1\), we obtain on the left hand side in (10)

\[
(L_p(\Omega), W^m_p(\Omega) \cap L_{p,v})_{\theta,p} = W^{m\theta}_p(\Omega) \cap \{ u : D^\alpha u |_{\partial \Omega} = 0 ; \ |x| \leq l \}.
\]

There we used the facts that \( W^m_p(\Omega) \cap L_{p,v} = W^{m(1)}_p(\Omega) \) and \(-1 - lp)/vp > (1 + lp)/mp \) and therefore Proposition (a) and Remark 1 (ii), (iii) are applicable. Further, \( \theta v < -(1/p) - l \). Thus, using (8), (9) and Remark 2, we conclude that the right hand side of (10) equals the left hand side. This proves (10) not only for (b) (i) but for (b) (ii) as well. Now, let the assumptions of (b) (i) be satisfied and let

\[
\frac{-1 - rp}{vp} \leq \theta < \frac{1 + (r + 1)p}{mp}, \ r = 0, ..., l - 1.
\]

For the left hand side of (10) we obtain because of Lemma (a) and (b):

\[
(L_p(\Omega), W^m_p(\Omega) \cap L_{p,v})_{\theta,p} = (L_p(\Omega), W^{m(1)}_p(\Omega))_{\theta,p} =
\]

\[
= W^m_p(\Omega) \cap \{ u : D^\alpha u |_{\partial \Omega} = 0 \ \forall x, \ 0 \leq |x| \leq r \}.
\]

For \( v \) we have

\[
v < \frac{-(1 + kp)m}{(1 + (k + 1)p)} < \frac{(1 + (k - 1)p)m}{1 + kp} < ... < \frac{-m}{1 + p}
\]

and so

\[
\frac{-1 - rp}{vp} < \frac{1 + (r + 1)p}{m}, \ r = 0, ..., k_0.
\]

The relations \( m\theta > r + (1/p) \) and \( \theta v \leq -(1/p) - r \), \( r = 0, ..., k_0 \), are always satisfied. Thus, using (8), (9) and Remarks 2 and 3, we conclude that the left hand side of (10) equals the right hand side.
Third step: We prove (c): Let
\[ \theta = \frac{1 + rp}{mp} \quad \text{with} \quad r = 0, \ldots, k_0 \]
in the case (b) (i) and \( r = 0 \) in the case (b) (ii). Using the statement (a) of Lemma and (7), we obtain for the left hand side in (10):
\[ (L_p(\Omega), W_p^m(\Omega) \cap L_p, u)_{\theta, p} = (L_p(\Omega), W_p^{m(1)}(\Omega))_{\theta, p} = \]
\[ = W_p^{m\theta}(\Omega) \cap \{ u : D^s u|_{\partial \Omega} = 0 ; \; |x| < r ; \; D^s u \in L_{p,-1/p} ; \; |x| = r \} . \]

With the aid of (8) and (9) we obtain on the right hand side of (10) the space \( W_p^{m\theta}(\Omega) \cap L_{p, -1/p} \), where \( \theta v > -(1/p) - r \); thus, this space is different from \( (L_p(\Omega), W_p^{m(1)}(\Omega))_{\theta, p} \).

Now, let
\[ \frac{1 + rp}{mp} < \theta < -\frac{1 - rp}{vp}, \quad r = 0, \ldots, k_0, \]
in the case (b) (i).

On the left hand side in (10) we obtain as above
\[ (L_p(\Omega), W_p^{m(1)}(\Omega))_{\theta, p} = W_p^{m\theta}(\Omega) \cap \{ u : D^s u|_{\partial \Omega} = 0 ; \; |x| \leq r \} . \]

Because of \( \theta v > -(1/p) - r \) and \( m\theta > r + (1/p) \) this space differs from the space on the right hand side.

Now, let
\[ \frac{1 + (k_0 + 1)p}{mp} < \theta < \frac{-1 - lp}{mp} \]
in the case (b) (i), and
\[ \frac{1}{mp} < \theta < \frac{-1 - lp}{mp} \]
in the case (b) (ii).

Then \( \theta v > -(1/p + (k_0 + 1)) \) in the case (b) (i) and \( \theta v > -1/p \) in the case (b) (ii). The space arising on the left hand side of (10) in the case (b) (i) is a subspace of
\[ W_p^{m\theta}(\Omega) \cap \{ u : D^s u|_{\partial \Omega} = 0 ; \; |x| \leq k_0 + 1 \} \]
while in the case (b) (ii) it is a subspace of
\[ W_p^{m\theta}(\Omega) \cap \{ u : u|_{\partial \Omega} = 0 \} . \]
The spaces arising on the right hand side of (10) are not subspaces of this space, which proves (c).
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