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STABILITY AT CONSTANTLY ACTING DISTURBANCES OF ABSTRACT DIFFERENTIAL EQUATIONS WITH THE RIGHT-HAND SIDES SMOOTH IN THE TIME VARIABLE

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1. INTRODUCTION

In the paper [1] the Lyapunov stability and the stability at constantly acting disturbances of solutions of the differential equation

$$(1.1)\qquad \mathscr{L}\, u(t) \equiv u^{(n)}(t) + a_1(A)\, u^{(n-1)}(t) + \ldots + a_n(A)\, u(t) = \\ = F(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$$

under the assumption $F \in \mathscr{C}(\mathscr{D}(u), \mathscr{D}(A^{1/n}))$ was investigated. (For simplicity we shall write $F(t, u(t))$ instead of $F(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$.)

The Lyapunov stability of solutions of the equation (1.1) under the assumption $F \in \mathscr{C}^{(1)}(\mathscr{D}(u), H)$ was studied in [2]. The aim of this paper is to prove three theorems about the stability at constantly acting disturbances of solutions of the equation (1.1) in the case $F \in \mathscr{C}^{(1)}(\mathscr{D}(u), H)$. We shall use all notations and conventions introduced in [1], [2]. The most important of them are: the operator A is selfadjoint, strictly positive (i.e. $\inf \{s \mid s \in \sigma(A)\} = \delta > 0$) with $\mathscr{D}(A) \subseteq H$, H being the Hilbert space with the norm $\|\cdot\|$, the operator-functions a_i satisfy the condition (1.1.1) from [1] (roughly speaking $\|a_i(A)\, A^{-i/n}\varphi\| \leqq C_0^* \|\varphi\|$, for $i = 1, \ldots, n$, $\varphi \in H$).

Let $v \in \mathscr{D}(\mathscr{L})$, $R : \mathscr{D}(R) \to H$. We shall write $R \in \mathscr{C}^{(1)}(\mathscr{D}(u), \mathscr{B}(v, r, H))$ where $r > 0$, if $\mathscr{D}(R) \subseteq \{(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) \mid u \in \mathscr{D}(\mathscr{L}),\ t \in \mathscr{D}(u)$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, $|||u(t) - v(t)||| \leqq r\}$ and if for all $u \in \mathscr{D}(\mathscr{L})$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ the functions $\|R(t, u(t))\|$, $\|R'(t, u(t))\|$ are continuous functions of the variable t for $t \in \{t \in \mathscr{D}(u) \mid |||u(t) - v(t)||| \leqq r\}$. (Here R' means the total derivative of the function R with respect to the variable t and $R(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$ is abbreviated to $R(t, u(t))$.)

Let $v : \mathscr{D}(v) \to H$ be a solution of the equation (1.1). We shall deal with the so called *disturbed equation*

$$(1.2)\qquad \mathscr{L}\, u(t) = F(t, u(t)) + R(t, u(t)) \quad \text{where} \quad F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)\,, \\ R \in \mathscr{C}^{(1)}(\mathscr{D}(u), \mathscr{B}(v, r, H))\,.$$

Let us note that a solution of (1.2) must satisfy the relations $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, $|||u(t) - v(t)||| \leqq r$ for $t \in \mathscr{D}(u)$ (see [1], Section 1.1).

Let us recall Definition 1.1.3 from [1].

Definition 1.1. Let $v : \mathscr{D}(v) \to H$ be a solution of the equation (1.1), $r > 0$. We say that v *is uniformly stable at constantly acting disturbances with respect to the norms* $|||\cdot|||$, $\|\cdot\|_D$ if for any $\eta \in (0, r]$ there exist positive numbers η_0, η_D such that the implication

$$(1.3) \quad \{|||u(t_0) - v(t_0)||| \leqq \eta_0, \ \|R(t, u(t))\|_D \leqq \eta_D \text{ for such } t \in \mathscr{D}(u) \text{ for which } |||u(t) - v(t)||| \leqq \eta\} \Rightarrow \{|||u(t) - v(t)||| \leqq \eta \text{ for all } t \in \mathscr{D}(u)\}$$

holds for every $t_0 \in \mathscr{D}(v)$ and for all solutions u of the equation (1.2) for which $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u)$.

Remark 1.1. In this paper we shall use the norms

$$|||u(t)||| = [\sum_{i=0}^{n-1} \|A^{(n-i)/n} u^{(i)}(t)\|^2]^{1/2},$$

$$\|R(t, u(t))\|_D = \max(\|R(t, u(t))\|, \|R'(t, u(t))\|).$$

Remark 1.2. We shall often use the notions *the type of the operator* $\mathscr{L}$, *the stable operator* $\mathscr{L}$, *the exponentially stable operator* $\mathscr{L}$. They were defined in [1] (Definitions 1.2.1 and 1.4.1). The constants $C(\mathscr{L})$ and C_5^*, C_6^*, which we shall use, were introduced in [1] (Definition 1.2.1) and in [2] (Lemma 2.4), respectively.

2. STABILITY AT CONSTANTLY ACTING DISTURBANCES IN THE CASE OF THE EXPONENTIALLY STABLE OPERATOR

Theorem 2.1. *Let $v : \mathscr{D}(v) \to H$ be a maximal solution of the equation (1.1). Let the operator $\mathscr{L}$ be of the type $\omega < 0$, $F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)$, $R \in \mathscr{C}^{(1)}(\mathscr{D}(u), \mathscr{B}(v, r, H))$ and let the conditions (2.1), (2.2) be fulfilled.*

(2.1) *There exists a constant K^* such that $\|Ax\| \leqq K^*\|a_n(A)\, x\|$ for all $x \in \mathscr{D}(A)$.*

(2.2) *There exists numbers $K_1, K_2, K_3, R > 0$ such that if u is a solution of the equation $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t)) + R(t, u(t))$ and $t \in \mathscr{D}(u)$ is such that $|||u(t)||| \leqq R$ then $\|F(t, v(t) + u(t)) - F(t, v(t))\| \leqq K_1 \sum_{i=0}^{n-1} \|A^{(n-i-1)/n} u^{(i)}(t)\|$, $\|F'(t, v(t) + u(t)) - F'(t, v(t))\| \leqq K_2|||u(t)||| + K_3\|R(t, u(t))\|_D$.*

Finally, let $\omega + C_5^* K_2 + C(\mathscr{L})\, C_6^* K_1^2 n^{3/2} \delta^{-1/n} < 0$. Then the solution v is uniformly stable at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_D$.

Proof. It is easy to see (compare [1] (Theorem 2.1.2)) that it suffices to prove the uniform stability at constantly acting disturbances of the zero solution $O_{/\mathscr{D}(v)}$ of the equation

$$(1) \qquad \mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))\,.$$

Let $\eta \in (0, r]$ be given. Without loss of generality we can suppose $\eta \leqq R$. Let us take a number $h > 0$ such that

$$(C_3^* + C_5^* K_1 n^{1/2} \delta^{-1/n} + C_2^* C_6^* K_1 n^{3/2} \delta^{-1/n})\, e^{(\omega + C_5^* K_2 + C(\mathscr{L}) C_6^* K_1^2 n^{3/2} \delta^{-1/n})h} < 1\,,$$

(the constant C_2^* was introduced in [1] (Lemma 1.3.3), $C_3^* = nC_2^*$) and choose numbers $\eta_0 \in (0, \eta/2]$, $\eta_D > 0$ fulfilling the inequalities

$$(2) \quad [C_3^* + C_5^* K_1 n^{1/2} \delta^{-1/n} + C_2^* C_6^* K_1 n^{3/2} \delta^{-1/n}]\, \eta_0 + \\ + [C_5^* + (C(\mathscr{L})\, C_6^* K_1 n |\omega|^{-1} + C_6^* + C_5^* K_3 |\omega|^{-1} + C_5^* |\omega|^{-1})\, e^{-\omega h}]\, \eta_D \leqq \\ \leqq \eta/2,$$

$$(3) \quad \{[C_3^* + C_5^* K_1 n^{1/2} \delta^{-1/n} + C_2^* C_6^* K_1 n^{3/2} \delta^{-1/n}]\, \eta_0 + \\ + [C_5^* + (C(\mathscr{L})\, C_6^* K_1 n |\omega|^{-1} + C_6^* + C_5^* K_3 |\omega|^{-1} + \\ + C_5^* |\omega|^{-1})\, e^{-\omega h}]\, \eta_D\}\, e^{(\omega + C_5^* K_2 + C(\mathscr{L}) C_6^* K_1^2 n^{3/2} \delta^{-1/n})h} \leqq \eta_0\,.$$

Let $t_0 \in \mathscr{D}(v)$, and let $u : \mathscr{D}(u) \to H$, $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u)$ be a solution of the equation

$$(4) \qquad \mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t)) + R(t, u(t))\,.$$

Then by [2] (Lemma 2.4):

$$(5) \quad |||u(t)||| \leqq C_3^* |||u(t_0)|||\, e^{\omega(t - t_0)} + C_5^* \{\|F(t_0, v(t_0) + u(t_0)) - F(t_0, v(t_0))\| + \\ + \|R(t_0, u(t_0))\|\}\, e^{\omega(t - t_0)} + C_6^* \{\|F(t, v(t) + u(t)) - F(t, v(t))\| + \\ + \|R(t, u(t))\|\} + C_5^* \int_{t_0}^{t} e^{\omega(t - \tau)} \{\|F'(\tau, v(\tau) + u(\tau)) - F'(\tau, v(\tau))\| + \\ + \|R'(\tau, u(\tau))\|\}\, d\tau \text{ for } t \in \mathscr{D}(u).$$

Now we shall prove the validity of the implication (1.3) (of course with $v \equiv O_{/\mathscr{D}(v)}$).

Let us suppose

(6) there exists a number $\tilde{h} \leqq h$ such that $[t_0, t_0 + \tilde{h}] \subseteq \mathscr{D}(u)$, $|||u(\tau)||| < \eta$ for $\tau \in [t_0, t_0 + \tilde{h})$, $|||u(t_0 + \tilde{h})||| = \eta$.

Then using (5), (2.2) and the relation $\eta \leqq R$ we obtain

$$(7) \quad |||u(t)||| \leqq C_3^* \eta_0 e^{\omega(t - t_0)} + \\ + C_5^* K_1 \sum_{i=0}^{n-1} \|A^{(n-i-1)/n} u^{(i)}(t_0)\|\, e^{\omega(t - t_0)} + C_5^* \eta_D e^{\omega(t - t_0)} +$$

$$+ C_6^* K_1 \sum_{i=0}^{n-1} \|A^{(n-i-1)/n} u^{(i)}(t)\| + C_6^* \eta_D +$$

$$+ C_5^* \int_{t_0}^{t} e^{\omega(t-\tau)} [K_2 |||u(\tau)||| + K_3 \|R(\tau, u(\tau))\|_D + \eta_D] \, d\tau,$$

for $t \in [t_0, t_0 + \tilde{h}]$.

Similarly to [1] (Theorem 2.1.1), we have

$$A^{-1/n} u(t) = \sum_{j=0}^{n-1} m_j(t; t_0, A) A^{-1/n} \varphi_j +$$

$$+ \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A) A^{-1/n} [F(\tau, v(\tau) + u(\tau)) - F(\tau, v(\tau)) + R(\tau, u(\tau))] \, d\tau$$

and so with help of Theorem 1.2.1 and Remark 1.3.1 from [1] we get according to (2.2)

$$(8) \quad \|A^{(n-i-1)/n} u^{(i)}(t)\| \leqq C_2^* \sum_{j=0}^{n-1} \|A^{(n-j-1)/n} \varphi_j\| e^{\omega(t-t_0)} +$$

$$+ C(\mathscr{L}) K_1 \int_{t_0}^{t} e^{\omega(t-\tau)} \sum_{j=0}^{n-1} \|A^{(n-j-1)/n} u^{(j)}(\tau)\| \, d\tau +$$

$$+ C(\mathscr{L}) \eta_D \int_{t_0}^{t} e^{\omega(t-\tau)} \, d\tau \text{ for } t \in [t_0, t_0 + \tilde{h}], \; i = 0, \ldots, n-1.$$

The relations (7), (8) together with the relation

$$\int_{t_0}^{t} e^{\omega(t-\tau)} \, d\tau = -\frac{1}{\omega} (1 - e^{\omega(t-t_0)}) < -\frac{1}{\omega} = \frac{1}{|\omega|}$$

and with Lemma 2.5 from [2] give

$$|||u(t)||| \leqq (C_3^* + C_5^* K_1 n^{1/2} \delta^{-1/n} + C_2^* C_6^* K_1 n^{3/2} \delta^{-1/n}) \eta_0 e^{\omega(t-t_0)} +$$

$$+ (C_5^* K_2 + C(\mathscr{L}) C_6^* K_1^2 n^{3/2} \delta^{-1/n}) \int_{t_0}^{t} e^{\omega(t-\tau)} |||u(\tau)||| \, d\tau +$$

$$+ [C_5^* + (C(\mathscr{L}) C_6^* K_1 n |\omega|^{-1} + C_6^* + C_5^* K_3 |\omega|^{-1} + C_5^* |\omega|^{-1}) e^{-\omega h}] \,.$$

$$. \, \eta_D e^{\omega(t-t_0)} \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}]$$

and so using [1] (Theorem 2.1.3) we can conclude

$$(9) \quad |||u(t)||| \leqq \{[C_3^* + C_5^* K_1 n^{1/2} \delta^{-1/n} + C_2^* C_6^* K_1 n^{3/2} \delta^{-1/n}] \eta_0 +$$

$$+ [C_5^* + (C(\mathscr{L}) C_6^* K_1 n |\omega|^{-1} + C_6^* + C_5^* K_3 |\omega|^{-1} + C_5^* |\omega|^{-1}) e^{-\omega h}] \eta_D\} \,.$$

$$. \, e^{(\omega + C_5^* K_2 + C(\mathscr{L}) C_6^* K_1^2 n^{3/2} \delta^{-1/n})(t-t_0)} \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}].$$

The relations (2), (9) together with the inequality
$\omega + C_5^* K_2 + C(\mathscr{L})\, C_6^* K_1^2 n^{3/2} \delta^{-1/n} < 0$ imply $|||u(t_0 + \tilde{h})||| \leqq \eta/2 < \eta$, which contradicts (6). So we have proved

$$(10) \qquad |||u(t)||| \leqq \eta \quad \text{for} \quad t \in [t_0, t_0 + h] \cap \mathscr{D}(u)\,.$$

If $t_0 + h \in \mathscr{D}(u)$ then by (3), (9)

$$(11) \qquad |||u(t_0 + h)||| \leqq \eta_0\,.$$

The validity of the implication (1.3) follows now from (10), (11). Let us find a natural number k and a number $s \in [0, h)$ such that $t = t_0 + kh + s$ to every $t \in \mathscr{D}(u)$. Then using k-times the relation (11) we see that $|||u(t_0 + kh)||| \leqq \eta_0$ and so by (10) $|||u(t)||| = |||u(t_0 + kh + s)||| \leqq \eta$. This proves the implication (1.3). The theorem is proved.

Theorem 2.2. *Let $v : \mathscr{D}(v) \to H$ be a maximal solution of the equation* (1.1). *Let the operator $\mathscr{L}$ be of the type $\omega < 0$, $F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)$, $R \in \mathscr{C}^{(1)}(\mathscr{D}(u), \mathscr{B}(v, r, H))$ and let the conditions* (2.1), (2.3), (2.4) *be fulfilled.*

(2.3) *$F(t, v(t) + u(t)) = F(t, v(t)) + F_L(t, u(t)) + F_N(t, u(t))$ for $u \in U$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and $t \in \mathscr{D}(u)$, where $F_L, F_N \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)$.*

(2.4) *There exist numbers $C_1, C_2, C_3, C_4, C_5, C_6$, $R_1 > 0$, $\nu_1 > 0$, $\nu_2 > 0$ such that if u is a solution of the equation $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t)) + R(t, u(t))$ and $t \in \mathscr{D}(u)$ is such that $|||u(t)||| \leqq R_1$ then*

$$\|F_L(t, u(t))\| \leqq C_1 \sum_{i=0}^{n-1} \|A^{(n-i-1)/n}\, u^{(i)}(t)\|,$$

$$\|F_N(t, u(t))\| \leqq C_2 |||u(t)|||^{\nu_1} \sum_{i=0}^{n-1} \|A^{(n-i-1)/n}\, u^{(i)}(t)\|,$$

$$\|F'_L(t, u(t))\| \leqq C_3 |||u(t)||| + C_4 \|R(t, u(t))\|_D,$$

$$\|F'_N(t, u(t))\| \leqq C_5 |||u(t)|||^{1+\nu_2} + C_6 \|R(t, u(t))\|_D.$$

Finally, let $\omega + C_5^* C_3 + C(\mathscr{L})\, C_6^* C_1^2 n^{3/2} \delta^{-1/n} < 0$. Then the solution v is uniformly stable at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_D$.

Proof. Let us choose a number $R \in (0, R_1]$ so small that

$$(1) \qquad \omega + C_5^*(C_3 + C_5 R^{\nu_2}) + C(\mathscr{L})\, C_6^*(C_1 + C_2 R^{\nu_1})^2\, n^{3/2} \delta^{-1/n} < 0\,.$$

Then if u is a solution of the equation (1.2) and $t \in \mathscr{D}(u)$ fulfils $|||u(t)||| \leqq R$ we obtain according to (2.3), (2.4)

$$(2) \qquad \|F(t, v(t) + u(t)) - F(t, v(t))\| \leqq (C_1 + C_2 R^{\nu_1}) \sum_{i=0}^{n-1} \|A^{(n-i-1)/n}\, u^{(i)}(t)\|\,,$$

(3) $$\|F'(t, v(t) + u(t)) - F'(t, v(t))\| \leqq$$
$$\leqq (C_3 + C_5 R^{\nu_2}) |||u(t)||| + (C_4 + C_6) \|R(t, u(t))\|_D .$$

The theorem now follows from (1), (2), (3) with help of Theorem 2.1.

3. STABILITY AT CONSTANTLY ACTING DISTURBANCES IN THE CASE OF THE STABLE OPERATOR

Theorem 3.1. *Let $v : \mathscr{D}(v) \to H$ be a maximal solution of the equation* (1.1). *Let the operator $\mathscr{L}$ be of the type* 0, $F \in \mathscr{C}^{(1)}(\mathscr{D}(v), H)$ *and let the conditions* (2.1), (2.3), (3.1), (3.2) *be fulfilled.*

(3.1) *There exist constants $K_1, K_2, K_3, K_4, K_5, K_6$, $R > 0$, $\nu_1 > 0$, $\nu_2 > 0$ such that if u_L solves the equation $\mathscr{L} u(t) = F_L(t, u(t))$, u_D solves the equation $\mathscr{L} u(t) = F_L(t, u(t)) + F_N(t, u(t)) + R(t, u(t))$, $R \in \mathscr{C}^{(1)}(\mathscr{D}(u), \mathscr{B}(v, r, H))$ and if $\mathscr{D}(u_D) \subseteq \mathscr{D}(u_L) \subseteq \mathscr{D}(v)$ then*

(i) $\|F_L(t, u_D(t))\| \leqq K_1 |||u_D(t)|||$ *for* $t \in \mathscr{D}(u_D)$ *such that* $|||u_D(t)||| \leqq R$, $\|F'_L(t, u_D(t)) - F'_L(t, u_L(t))\| \leqq K_2 |||u_D(t) - u_L(t)||| + K_3 \|R(t, u_D(t))\|_D$ *for* $t \in \mathscr{D}(u_D)$ *such that* $|||u_L(t)||| \leqq R$, $|||u_D(t)||| \leqq R$,

(ii) $\|F_N(t, u_D(t))\| \leqq K_4 |||u_D(t)|||^{1+\nu_1}$, $\|F'_N(t, u_D(t))\| \leqq K_5 |||u_D(t)|||^{1+\nu_2} + K_6 \|R(t, u_D(t))\|_D$ *for* $t \in \mathscr{D}(u_D)$ *such that* $|||u_D(t)||| \leqq R$.

(3.2) *There exists a number $\varkappa > 0$ such that if $\varphi_i \in \mathscr{D}(A^{(n-i)/n})$, $(i = 0, \ldots, n-1)$, $[\sum_{i=0}^{n-1} \|A^{(n-i)/n} \varphi_i\|^2]^{1/2} \leqq \varkappa$ and $t_0 \in \mathscr{D}(v)$ then there exists a maximal solution u of the equation $\mathscr{L} u(t) = F_L(t, u(t))$ which fulfils the initial conditions $u^{(i)}(t_0) = \varphi_i$, $(i = 0, \ldots, n-1)$.*

Let $F_L(t, O_{/I}) = 0$, $F'_L(t, O_{/I}) = 0$ for every $I \subseteq \mathscr{D}(v)$. Further, let the zero solution $O_{/\mathscr{D}(v)}$ of the equation

(3.3) $$\mathscr{L} u(t) = F_L(t, u(t))$$

be uniformly exponentially stable with respect to the norm $|||\cdot|||$. Then the solution v is uniformly stable at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_D$.

Proof. Clearly (see [1] (Theorem 2.1.2)) it suffices to prove the uniform stability at constantly acting disturbances of the zero solution $O_{/\mathscr{D}(v)}$ of the equation

(1) $$\mathscr{L} u(t) = F_L(t, u(t)) + F_N(t, u(t)) .$$

Remember that the uniform exponential stability of the solution $O_{/\mathscr{D}(v)}$ of the equation (3.3) means

(2) there exist positive numbers C, α, ϱ such that if u_L is a solution of the equation (3.3) then $|||u_L(t_0)||| \leqq \varrho \Rightarrow |||u_L(t)||| \leqq Ce^{-\alpha(t-t_0)}|||u_L(t_0)|||$ for $t \in \mathscr{D}(u_L)$ whenever $t_0 = \min_{t \in \mathscr{D}(u_L)} t \in \mathscr{D}(v)$.

Let $\eta \in (0, r]$ be given. Without loss of generality we may suppose $\eta \leqq \min(R, \varkappa, \varrho)$. Denote $B = K^* + C_7^*$ (the constant C_7^* was introduced in [2] (Lemma 3.2)). Let $h > 0$ be such that $Ce^{-\alpha h} < 1$ and a number $R_1 \in (0, \eta]$ such that

$$Ce^{-\alpha h} + BK_5R_1^{\nu_2}h(C_3^* + BK_1)\, e^{B(2K_2 + K_5\eta^{\nu_2})h} < 1\,.$$

Finally, let us choose numbers $\eta_0 \in (0, R_1/2]$, $\eta_D > 0$ such that $C\eta_0 \leqq R$ and

(3) $[(C_3^* + BK_1 + BK_4\eta_0^{\nu_1})\,\eta_0 + B(1 + h(K_3 + K_6 + 1))\,\eta_D]\, e^{B(K_2 + K_5\eta^{\nu_2})h} < R_1,$

(4) $[Ce^{-\alpha h} + BK_4\eta_0^{\nu_1}e^{BK_2h} + BK_5R_1^{\nu_2}h(C_3^* + BK_1 + BK_4\eta_0^{\nu_1})\, e^{B(2K_2 + K_5\eta^{\nu_2})h}]\,\eta_0 +$

$+ [B + B(K_3 + K_6 + 1)\,h + B^2K_5R_1^{\nu_2}h(1 + h(K_3 + K_6 + 1))\,.$

$.\, e^{B(K_2 + K_5\eta^{\nu_2})h}]\, e^{BK_2h}\eta_D \leqq \eta_0.$

Now let $t_0 \in \mathscr{D}(v)$ and let $u_D : \mathscr{D}(u_D) \to H$ be a solution of the equation

(5) $$\mathscr{L}\, u(t) = F_L(t, u(t)) + F_N(t, u(t)) + R(t, u(t))\,,$$

such that $\mathscr{D}(u_D) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u_D)$. To prove the theorem, we have to prove the implication (1.3) (of course with $v \equiv O_{/\mathscr{D}(v)}$). Let us suppose

(6) there exists a number $\tilde{h} < h$ such that $[t_0, t_0 + \tilde{h}] \subseteq \mathscr{D}(u_D)$, $|||u_D(\tau)||| < R_1$ for $\tau \in [t_0, t_0 + \tilde{h})$,
$|||u_D(t_0 + \tilde{h})||| = R_1$.

Then using [2] (Lemma 3.2) and (3.1) we get according to (6):

$$|||u_D(t)||| \leqq C_3^*|||u_D(t_0)||| + B(\|F_L(t_0, u_D(t_0))\| +$$
$$+ \|F_N(t_0, u_D(t_0))\| + \|R(t_0, u_D(t_0))\|) +$$
$$+ B\int_{t_0}^{t} (\|F'_L(\tau, u_D(\tau))\| + \|F'_N(\tau, u_D(\tau))\| + \|R'(\tau, u_D(\tau))\|)\, d\tau \leqq$$
$$\leqq C_3^*\eta_0 + B(K_1\eta_0 + K_4\eta_0^{1+\nu_1} + \eta_D) +$$
$$+ B\int_{t_0}^{t} (K_2|||u_D(\tau)||| + K_3\eta_D + K_5|||u_D(\tau)|||^{1+\nu_2} + K_6\eta_D + \eta_D)\, d\tau \leqq$$
$$\leqq (C_3^* + BK_1 + BK_4\eta_0^{\nu_1})\,\eta_0 + B(1 + h(K_3 + K_6 + 1))\,\eta_D +$$
$$+ B(K_2 + K_5\eta^{\nu_2})\int_{t_0}^{t} |||u_D(\tau)|||\, d\tau \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}]$$

and thus (see [1] (Theorem 2.1.3)) with help of $\tilde{h} < h$ and (3) we obtain

(7) $$|||u_D(t)||| \leqq [(C_3^* + BK_1 + BK_4\eta_0^{\nu_1})\,\eta_0 + B(1 + h(K_3 + K_6 + 1))\,\eta_D]\,.$$
$$.\, e^{B(K_2 + K_5\eta^{\nu_2})h} < R_1 \text{ for } t \in [t_0, t_0 + \tilde{h}].$$

But this contradicts (6). So we have proved

(8) $$|||u_D(t)||| \leqq R_1 \leqq \eta \quad \text{for} \quad t \in [t_0, t_0 + h] \cap \mathscr{D}(u_D)\,.$$

By (3.2) there exists a maximal solution u_L of the equation (3.3) fulfilling the same initial conditions as the solution u_D. With help of [2] (Lemma 3.2) we obtain

$$u_D(t) - u_L(t) = M(t; t_0, A)\,[F_N(t_0, u_D(t_0)) + R(t_0, u_D(t_0))] + \\ + \int_{t_0}^{t} M(t + t_0 - \tau; t_0, A)\,[F'_L(\tau, u_D(\tau)) - F'_L(\tau, u_L(\tau)) + \\ + F'_N(\tau, u_D(\tau)) + R'(\tau, u_D(\tau))]\mathrm{d}\tau \quad \text{for} \quad t \in \mathscr{D}(u_D)$$

and thus using [2] (Lemma 3.2) again and (3.1), (7), (8) (we can use the relation (3.1) in virtue of (8) and the inequalities

$$\eta \leqq R\,, \quad |||u_L(t)||| \leqq Ce^{-\alpha(t-t_0)}|||u_L(t_0)||| \leqq C\eta_0 \leqq R)\,,$$

we get

$$|||u_D(t) - u_L(t)||| \leqq B(\|F_N(t_0, u_D(t_0))\| + \|R(t_0, u_D(t_0))\|) + \\ + B\int_{t_0}^{t} [\|F'_L(\tau, u_D(\tau)) - F'_L(\tau, u_L(\tau))\| + \|F'_N(\tau, u_D(\tau))\| + \\ + \|R'(\tau, u_D(\tau))\|]\,\mathrm{d}\tau \leqq B(K_4\eta_0^{1+\nu_1} + \eta_D) + \\ + B\int_{t_0}^{t} (K_2|||u_D(\tau) - u_L(\tau)||| + K_3\eta_D + K_5|||u_D(\tau)|||^{1+\nu_2} + \\ + K_6\eta_D + \eta_D)\,\mathrm{d}\tau \leqq [BK_4\eta_0^{\nu_1} + BK_5R_1^{\nu_2}h(C_3^* + BK_1 + BK_4\eta_0^{\nu_1})\,. \\ .\, e^{B(K_2 + K_5\eta^{\nu_2})h}]\,\eta_0 + [B + B(K_3 + K_6 + 1)\,h + \\ + B^2K_5R_1^{\nu_2}h(1 + h(K_3 + K_6 + 1))\,e^{B(K_2 + K_5\eta^{\nu_2})h}]\,\eta_D + \\ + BK_2\int_{t_0}^{t} |||u_D(\tau) - u_L(\tau)|||\,\mathrm{d}\tau$$

for $t \in [t_0, t_0 + h] \cap \mathscr{D}(u_D)$. This with help of Gronwall's lemma (see [1] (Theorem 2.1.3)) and with help of (2) yields: if $t_0 + h \in \mathscr{D}(u_D)$ then

$$|||u_D(t_0 + h)||| \leqq |||u_L(t_0 + h)||| + |||u_D(t_0 + h) - u_L(t_0 + h)||| \leqq \\ \leqq [Ce^{-\alpha h} + BK_4\eta_0^{\nu_1}e^{BK_2h} + BK_5R_1^{\nu_2}h(C_3^* + BK_1 + BK_4\eta_0^{\nu_1})\,. \\ .\, e^{B(2K_2 + K_5\eta^{\nu_2})h}]\,\eta_0 + [B + B(K_3 + K_6 + 1)\,h + \\ + B^2K_5R_1^{\nu_2}h(1 + h(K_3 + K_6 + 1))\,e^{B(K_2 + K_5\eta^{\nu_2})h}]\,e^{BK_2h}\eta_D$$

and so by (4) we can conclude

$$(9) \qquad |||u_D(t_0 + h)||| \leqq \eta_0 \quad \text{if} \quad t_0 + h \in \mathcal{D}(u_D)\,.$$

Now the relations (8), (9) prove the implication (1.3). (The proof is the same as in the corresponding part of proof of Theorem 2.1.) The theorem is proved.

Remark 3.1. We can consider more general disturbances than in this paper (see [3]). For example, theorems analogous to Theorems 2.1, 2.2, 3.1 in the case $R(t, u(t)) = R_1(t, u(t)) + R_2(t, u(t))$, where $R_1 \in \mathscr{C}^{(1)}(\mathcal{D}(u), \mathscr{B}(v, r_1, H))$, $R_2 \in \mathscr{C}(\mathcal{D}(u), \mathscr{B}(v, r_2, \mathcal{D}(A^{1/n})))$, are introduced in Part 4 – Examples of [3].

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