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Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 3, 469–474

Persistent URL: <http://dml.cz/dmlcz/101761>

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HYPERGRAPHS AND INTERVALS

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(Received October 3, 1979)

1. In the present paper by a hypergraph (or a graph) we mean an ordered pair (V, \mathcal{E}) , where V is a finite nonempty set and \mathcal{E} is a set of subsets of V with the property that if $E \in \mathcal{E}$, then $|E| \geq 1$ (or $|E| = 2$, respectively); elements of V or \mathcal{E} are called *vertices* or *edges*, respectively. If $H = (V, \mathcal{E})$ is a hypergraph, then we shall write $V(H) = V$ and $\mathcal{E}(H) = \mathcal{E}$. Obviously, every graph is a hypergraph. Our concept of a graph is the same as that in [1] or [6], and our concept of a hypergraph is relative to the concept of a simple hypergraph in [2]. (Graph theoretical terms which we shall use without definitions can be found in [1] or [6].)

If \mathcal{E}_0 is a finite nonempty set of a finite nonempty sets, then we denote by $\langle \mathcal{E}_0 \rangle$ the hypergraph H_0 with

$$V(H_0) = \bigcup_{E_0 \in \mathcal{E}_0} E_0$$

and $\mathcal{E}(H_0) = \mathcal{E}_0$.

In the present paper the letters g, h, i, j, k, m , and n mean integers. We shall say that a sequence (v_1, \dots, v_n) , $n \geq 1$, is an arrangement, if for any g and h , $1 \leq g < h \leq n$, we have that $v_g \neq v_h$. An arrangement (v_1, \dots, v_n) is referred to as an arrangement on a set V if $V = \{v_1, \dots, v_n\}$. Let $\alpha = (v_1, \dots, v_n)$ be an arrangement on a set V ; we say that $X \subseteq V$ is an interval set in α if there exist i and k , $1 \leq i \leq k \leq n$, such that $X = \{v_j; i \leq j \leq k\}$; the set of all interval sets in α will be denoted by $\text{Int}(\alpha)$.

Let V be a finite nonempty set, and let A be a nonempty set of arrangements on V ; then we denote

$$\text{Int}(A) = \bigcap_{\alpha \in A} \text{Int}(\alpha).$$

Let H be a hypergraph. Following [9] we shall say that an arrangement π on $V(H)$ is a projectoidic arrangement on H if $\mathcal{E}(H) \subseteq \text{Int}(\pi)$. We shall say that H is a *projectoid* if there exists a projectoidic arrangement on H . Note that the terms “projectoidic” and “projectoid” have a relation to the term “projective” in the sense in which it is used in mathematical linguistics (see, for example, [8]).

Projectoids were investigated in [3], [4], [9], [10] and [11], but in [4] and [11] projectoids were studied by means of the matrix theory (as a class of $(0, 1)$ -matrices), and in [11] also as bipartite graphs. Various characterizations of projectoids are known. In this section the characterization given by the present author in [9] will be recorded.

Let H be a hypergraph. Following [9] we denote by $\Omega(H)$ the set of all sets $R \subseteq V(H)$ such that at least one of the following conditions holds:

- (1) there exists $v \in V(H)$ with the property that $R = \{v\}$;
- (2) $R \in \mathcal{E}(H)$;
- (3) there exist $R_1, R_2 \in \Omega(H)$ such that $R_1 \cap R_2 \neq \emptyset$ and $R = R_1 \cup R_2$.

The following theorem was proved in [9]:

Theorem 1. *A hypergraph H is a projectoid if and only if for every three $R_1, R_2, R_3 \in \Omega(H)$, the hypergraph $\langle \{R_1, R_2, R_3\} \rangle$ is a projectoid.*

2. Let H be a hypergraph. We denote by $\Pi(H)$ the set of projectoidic arrangements on H . This means that H is a projectoid if and only if $\Pi(H) \neq \emptyset$.

Let H be a projectoid. According to the definition, $\mathcal{E}(H) \subseteq \text{Int}(\Pi(H))$. In this section a characterization of the set $\text{Int}(\Pi(H))$ will be given.

Let H be a hypergraph. Consider arbitrary $X, Y \subseteq V(H)$; we shall write $X \sim Y$ if at least one of the sets $X \cap Y, X - Y$ and $Y - X$ is empty; on the other hand, we shall write $X \approx Y$ if the sets $X \cap Y, X - Y$ and $Y - X$ are nonempty. We denote by $\Sigma(H)$ the set of all sets $S \subseteq V(H)$ such that at least one of the following conditions holds:

- (0) $S = V(H)$;
- (1) there exists $v \in V(H)$ with the property that $S = \{v\}$;
- (2) $S \in \mathcal{E}(H)$;
- (3) there exist $S_1, S_2 \in \Sigma(H)$ such that $S_1 \cap S_2 \neq \emptyset$ and $S \in \{S_1 \cap S_2, S_1 \cup S_2\}$;
- (4) there exist $S', S'' \in \Sigma(H)$ such that $S' \approx S''$ and $S = S' - S''$.

Obviously, $\Omega(H) \subseteq \Sigma(H)$. We shall prove that if $\Pi(H) \neq \emptyset$, then $\Sigma(H) = \text{Int}(\Pi(H))$.

Let H be a hypergraph, and let $U \subseteq V(H)$. We say that U is free in H if for every $E \in \mathcal{E}(H)$, $E \sim U$.

Lemma 1. *Let H be a projectoid, $|V(H)| = n \geq 2$, let (v_1, \dots, v_n) be a projectoidic arrangement of H , and let $1 \leq j < n$. Then there exist i and k , $1 \leq i \leq j < k \leq n$, such that $\{v_i, \dots, v_j\}, \{v_{j+1}, \dots, v_k\} \in \Sigma(H)$, and $\{v_i, \dots, v_k\}$ is free in H .*

Proof. Denote

$$B = \{i_0; 1 \leq i_0 \leq j, \{v_{i_0}, \dots, v_j\} \in \Sigma(H)\} \quad \text{and}$$

$$B' = \{k_0; j + 1 \leq k_0 \leq n, \{v_{j+1}, \dots, v_{k_0}\} \in \Sigma(H)\}.$$

Since $\{v_j\}, \{v_{j+1}\} \in \Sigma(H)$, we have that $B \neq \emptyset \neq B'$. We denote by i and k the minimum integer in B and the maximum integer in B' , respectively. We wish to show that $\{v_i, \dots, v_k\}$ is free in H .

Assume, on the contrary, that $\{v_i, \dots, v_k\}$ is not free in H . Then there exists $E \in \mathcal{E}(H)$ such that $E \sim \{v_i, \dots, v_k\}$. Since (v_1, \dots, v_n) is a projectoidic arrangement on H , there exist f and g , $1 \leq f < g \leq n$, such that $E = \{v_f, \dots, v_g\}$. Since $\{v_f, \dots, v_g\} \sim \{v_i, \dots, v_k\}$, we have that either $f < i \leq g < k$ or $i < f \leq k < g$. We shall assume that $f < i \leq g < k$ (the latter case would be proved analogously).

First, let $g \leq j$; then $E \cap \{v_i, \dots, v_j\} \neq \emptyset$; we have that $\{v_f, \dots, v_j\} = E \cup \{v_i, \dots, v_j\}$. Next, let $g > j$; then $E \sim \{v_{j+1}, \dots, v_h\}$; we have that $\{v_f, \dots, v_j\} = E - \{v_{j+1}, \dots, v_h\}$. In both cases $\{v_f, \dots, v_j\} \in \Sigma(H)$ and thus $f = i$, which is a contradiction.

Thus $\{v_i, \dots, v_k\}$ is free in H , which completes the proof.

Lemma 2. *Let H be a projectoid, let π be a projectoidic arrangement of H , and let X be an interval set in π such that $X \notin \Sigma(H)$. Then there exists an interval set Z in π such that Z is free in H and $X \sim Z$.*

Proof. Denote $\pi = (v_1, \dots, v_n)$. Since $X \in \text{Int}(\pi)$, there exist g and j , $1 \leq g \leq j \leq n$, such that $X = \{v_g, \dots, v_j\}$. Since $X \notin \Sigma(H)$, we have that $|X| \neq n$. Without loss of generality we assume that $j < n$. According to Lemma 1, there exist i and k , $1 \leq i \leq j < k \leq n$, such that $\{v_i, \dots, v_k\}$ is free in H and $\{v_i, \dots, v_j\} \in \Sigma(H)$. Since $X \notin \Sigma(H)$, we have that $i \neq g$. If $i > g$, then we put $Z = \{v_i, \dots, v_k\}$, and the lemma is proved.

Now, assume that $i < g$. Then $1 < g$. According to Lemma 1, there exist f and h , $1 \leq f \leq g - 1 < h \leq n$, such that $\{v_f, \dots, v_h\}$ is free in H and $\{v_g, \dots, v_h\} \in \Sigma(H)$. Hence, $h \neq j$. If $h > j$, then $\{v_g, \dots, v_h\} \sim \{v_i, \dots, v_j\}$ and thus $\{v_g, \dots, v_j\} = \{v_g, \dots, v_h\} \cap \{v_i, \dots, v_j\} \in \Sigma(H)$, which is a contradiction. This means that $h < j$. We put $Z = \{v_f, \dots, v_h\}$ and the lemma is proved.

Theorem 2. *Let H be a projectoid. Then $\text{Int}(\Pi(H)) = \Sigma(H)$.*

Proof. Since H is a projectoid, we have that $\mathcal{E}(H) \subseteq \text{Int}(\Pi(H))$. It follows from the definition of $\Sigma(H)$ that $\Sigma(H) \subseteq \text{Int}(\Pi(H))$.

We now wish to prove that $\text{Int}(\Pi(H)) \subseteq \Sigma(H)$. Assume, on the contrary, that there exists $X \in \text{Int}(\Pi(H)) - \Sigma(H)$. Consider an arbitrary projectoidic arrangement $\pi = (v_1, \dots, v_n)$ on H . Since $X \in \text{Int}(\pi)$, there exist i and j , $1 \leq i \leq j \leq n$, such that $X = \{v_i, \dots, v_j\}$. Since $X \notin \Sigma(H)$, it follows from Lemma 2 that there exist g and h , $1 \leq g \leq h \leq n$, such that $\{v_g, \dots, v_h\}$ is free in H and $\{v_g, \dots, v_h\} \sim \{v_i, \dots, v_j\}$. Without loss of generality we assume that $g \leq i$. Then $g < i \leq h < j$. We denote by π' the sequence (v'_1, \dots, v'_n) , where $v'_m = v_m$ for $1 \leq m < g$ or $h < m \leq n$, and the subsequence (v'_g, \dots, v'_h) is identical with the sequence (v_h, \dots, v_g) . Since $\{v_g, \dots, v_h\}$

is free in H , we have that π' is a projectoidic arrangement on H . Obviously, $X \notin \text{Int}(\pi')$, which is a contradiction.

Thus the theorem is proved.

Corollary. *Let H_1 and H_2 be projectoids with $V(H_1) = V(H_2)$. Then $\Pi(H_1) \subseteq \Pi(H_2)$ if and only if $\Sigma(H_2) \subseteq \Sigma(H_1)$.*

Proof. If $\Pi(H_1) \subseteq \Pi(H_2)$, then $\text{Int}(\Pi(H_2)) \subseteq \text{Int}(\Pi(H_1))$ and therefore, according to Theorem 2, $\Sigma(H_2) \subseteq \Sigma(H_1)$.

Let $\Sigma(H_2) \subseteq \Sigma(H_1)$. Since $\mathcal{E}(H_2) \subseteq \Sigma(H_2)$ and $\Sigma(H_1) = \text{Int}(\Pi(H_1))$, we have that $\mathcal{E}(H_2) \subseteq \text{Int}(\pi)$ for each $\pi \in \Pi(H_1)$. Hence, $\Pi(H_1) \subseteq \Pi(H_2)$, which completes the proof.

3. A graph isomorphic to the intersection graph (in the sense of [6]) of a finite nonempty family of intervals in the real line is called an interval graph. Various characterizations of interval graphs can be found in [4], [5] and [7]. The following characterization of interval graphs is due to Lekkerkerker and Boland [7]:

(LB) A graph G is an interval graph if and only if (a) G contains no induced cycle of length ≥ 4 , and (b) for any three vertices v_1, v_2 and v_3 of G , there exist distinct $i, j, k \in \{1, 2, 3\}$ with the property that every $v_i - v_k$ path P in G contains a vertex v_p such that $\{v_j, v_p\} \in \mathcal{E}(G)$.

It is clear that a graph G is an interval graph if and only if there exists a projectoid H such that G is isomorphic to the intersection graph of $\mathcal{E}(H)$. Therefore, a necessary condition for a hypergraph H to be a projectoid is that the intersection graph of $\mathcal{E}(H)$ be an interval graph. However, examples showing that this condition is not sufficient can be easily found. On the other hand, the concept of projectoid can serve as a tool for characterizing interval graphs. In the language of the matrix theory such an approach was adopted by Fulkerson and Gross [4]. In the present section another approach will be suggested.

Let D be a digraph in the sense of [1] or [6]. We denote by $V(D)$ and $A(D)$ its vertex set and arc set, respectively. For every $v \in V(D)$, we denote

$$N(v, D) = \{w \in V(D); (v, w) \in A(D)\} \quad \text{and} \quad M(v, D) = N(v, D) \cup \{v\}.$$

We denote by M_D the hypergraph with $V(M_D) = V(D)$ and

$$\mathcal{E}(M_D) = \{M(v, D); v \in V(D)\}.$$

A hypergraph H is said to be an M -hypergraph of G if there exists a digraph D such that the underlying graph of D (see [1]) is identical with G , and M_D is identical with H .

Lemma 3. *Let G be an interval graph. Then at least one of its M -hypergraphs is a projectoid.*

Proof. Obviously, there exists a projectoid H such that G is isomorphic to the intersection graph of $\mathcal{E}(H)$. Consider a projectoidic arrangement (v_1, \dots, v_n) of H . For every $E \in \mathcal{E}(H)$, we denote $i(E) = \min \{i; v_i \in E\}$. Moreover, we denote by D the digraph with $V(D) = \mathcal{E}(H)$, such that $(E', E'') \in A(D)$ if and only if $v_{i(E'')} \in E'$ for any distinct $E', E'' \in \mathcal{E}(H)$. It is easy to see that the underlying graph of D is isomorphic to G . Denote $m = |\mathcal{E}(H)|$. Consider such an arrangement (E_1, \dots, E_m) that for any $j, k \in \{1, \dots, m\}$, $i(E_j) < i(E_k)$ implies $j < k$. It is clear that (E_1, \dots, E_m) is a projectoidic arrangement of M_D . Hence, M_D is a projectoid, which completes the proof.

Remark. Lemma 3 gives a necessary condition for a graph to be an interval graph, but this condition is not sufficient. For example, let D be a digraph with $n \geq 4$ vertices v_1, \dots, v_n such that

$$A(D) = \{(v_1, v_2), (v_1, v_3), \dots, (v_1, v_{n-1}), (v_n, v_2), (v_n, v_3), \dots, (v_n, v_{n-1})\}.$$

Since (v_1, \dots, v_n) is a projectoidic arrangement of M_D , we have that M_D is a projectoid. Obviously, the underlying graph of D is $K(2, n - 2)$, which is not an interval graph.

Lemma 4. *Let G be a graph with no induced cycle of length four. Assume that at least one of the M -hypergraphs of G is a projectoid. Then G is an interval graph.*

Proof. It follows from the assumption that there exists a digraph D such that the underlying graph of D is identical with G and M_D is a projectoid. Consider a projectoidic arrangement (v_1, \dots, v_n) of M_D .

First, let G contain an induced cycle C of length ≥ 5 . Then there exist i, j and k , $1 \leq i < j < k \leq n$, such that $v_i, v_j, v_k \in V(C)$, $\{v_i, v_k\} \in \mathcal{E}(C)$, and $\{v_i, v_j\}, \{v_j, v_k\} \notin \mathcal{E}(C)$. Without loss of generality we assume that $(v_i, v_k) \in A(D)$. Since C is an induced subgraph of G , we have that $\{v_i, v_j\} \notin \mathcal{E}(G)$. This implies that $v_j \notin M(v_i, D)$. Since $v_i, v_k \in M(v_i, D)$ and $i < j < k$, we have that (v_1, \dots, v_n) is not a projectoidic arrangement of M_D , which is a contradiction. This means that G contains no induced cycle of length > 3 .

Assume that G is not an interval graph. Since G contains no induced cycle of length > 3 , it follows from (LB) that there exist integers i, j, k , $1 \leq i < j < k \leq n$, and a $v_i - v_k$ path P in G such that for no $v \in V(P)$, $\{v_j, v\} \in \mathcal{E}(G)$. It is clear that there exist integers g and h , $1 \leq g < j < h \leq n$, such that $\{v_g, v_h\} \in \mathcal{E}(P)$. Without loss of generality we assume that $(v_g, v_h) \in A(D)$. It is obvious that $v_j \notin M(v_g, D)$. Therefore, (v_1, \dots, v_n) is not a projectoidic arrangement of M_D , which is a contradiction. This means that G is an interval graph, which completes the proof.

According to (LB), an interval graph contains no induced cycle of length four. Combining this observation with Lemmas 3 and 4, we find the following

Theorem 3. *A graph G is an interval graph if and only if G contains no cycle of length four and at least one of the M -hypergraphs of G is a projectoid.*

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