Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

ON THE LATTICE OF TORSION CLASSES OF LATTICE ORDERED GROUPS

JÁN JAKUBÍK, Košice

(Received June 20, 1979)

This note has been inspired by Jorge Martinez's paper [4] entitled 'Is the lattice of torsion classes algebraic?'. By using some results of the author's paper [2] it will be shown here that the answer is 'No'.

The standard denotations for lattice ordered groups will be used (cf. Conrad [1]). Let us recall some basic definitions concerning torsion classes.

For a lattice ordered group G we denote by $c(G)$ the system of all convex l-subgroups of G. The system $c(G)$ partially ordered by inclusion is a complete distributive lattice; the lattice operations in $c(G)$ will be denoted by $\wedge$, $\vee$. Let $\mathcal{G}$ be the class of all lattice ordered groups. The class containing the zero group $\{0\}$ only will be denoted by $\bar{0}$. Let Ord be the class of all ordinals.

A nonempty class C of lattice ordered groups is called *a torsion class* [3] if it has the following properties:

(a) Whenever $G \in C$ and $G_1 \in c(G)$, then $G_1 \in C$.

(b) If $G \in \mathcal{G}$ and $\{G_i\}_{i \in I} \subseteq C \cap c(G)$, then $\bigvee_{i \in I} G_i \in C$.

(c) The class C is closed with respect to homomorphisms.

Let Rad be the class of all torsion classes. The class Rad is partially ordered by inclusion. Then Rad is a complete lattice (cf. [3]); in fact, Rad is a proper class. If A is a subclass of Rad, then the symbols $\inf A$ and $\sup A$ are taken with respect to the complete lattice Rad.

Let C be a torsion class. C will be said to be *α-compact*, if, whenever $A \subseteq$ Rad and $C \leqq \sup A$, then there is $A_1 \subseteq A$ such that A_1 is finite and $C \leqq \sup A_1$. The torsion class C is called *β-compact*, if, whenever $A \subseteq$ Rad, $C \leqq \sup A$ and A is a set, then there is $A_1 \subseteq A$ such that A_1 is finite and $C \leqq \sup A_1$.

Let A be a nonempty subclass of Rad such that, whenever $\emptyset \neq A_1 \subseteq A$, then $\sup A_1 \in A$ and $\inf A_1 \in A$ (in other words, A is a closed sublattice of Rad). Consider the following conditions for A:

(α_1) For each $C \in A$ there exists $A_1 \subseteq A$ such that each element of A_1 is α-compact and $\sup A_1 = C$.

(β_1) For each $C \in A$ there exists a set $A_1 \subseteq A$ such that each element of A_1 is β-compact and $\sup A_1 = C$.

(γ_1) For each $C \in A$ there exists a set $A_1 \subseteq A$ such that each element of A_1 is α-compact and $\sup A_1 = C$.

A closed sublattice A of Rad is called *α-algebraic* (*β-algebraic* or *γ-algebraic*) if it fulfils the condition (α_1) (the condition (β_1) or (γ_1), respectively).

Let $G \in \mathcal{G}$. The intersection of all torsion classes containing G is a torsion class; it is said to be the *principal torsion class generated by* G. More generally, let $A_1 \subseteq \mathcal{G}$. The intersection A of all torsion classes B with $A_1 \subseteq B$ is a torsion class; it is called the *torsion class generated by* A_1 and we express this situation by writing $A = [A_1]$.

Let $C_1, C_2 \in$ Rad, $C_1 < C_2$. If the interval $[C_1, C_2]$ of the lattice Rad contains only the elements C_1 and C_2, then C_2 is said to be an *atom over* C_1. The class of all atoms over C_1 is denoted by $a(C_1)$.

Proposition 1. (Cf. [4], Proposition 2.4.) *The α-compact elements of* Rad *are those torsion classes which can be generated by one lattice ordered group with a strong order unit.*

Proposition 2. (Cf. [4], Proposition 2.2 (b).) *The lattice* Rad *is α-algebraic.*

By using the above terminology the question proposed by J. Martinez in [4] can be expressed by asking whether the lattice Rad is β-algebraic.

Let us investigate the condition (γ) first.

Proposition 3. (Cf. [2], Lemma 3.4 and Proposition 4.4.) *Let $A_1 \neq \emptyset$ be a set of principal torsion classes, $C = \sup A_1$. Then $a(C) \neq \emptyset$.*

Proposition 4. (Cf. [2], Theorem 5.6.) *There exists $C \in$ Rad with $C \neq \mathcal{G}$ such that $a(C) = \emptyset$.*

Proposition 5. *There is $C \in$ Rad with $C \neq \mathcal{G}$ such that the lattice $[\bar{0}, C]$ is not γ-algebraic.*

Proof. This follows from Propositions 1, 3 and 4.

Corollary 1. *The lattice* Rad *is not γ-algebraic.*

Let $\emptyset \neq P \subseteq \mathcal{G}$. Let us denote by

$S_c(P)$ – the class of all lattice ordered groups H' such that H' is a convex l-subgroup of some lattice ordered group $H \in P$;

$h(P)$ – the class of all homomorphic images of lattice ordered groups belonging to P;

$d(P)$ – the class of all lattice ordered groups that can be expressed as direct sums (= discrete direct products) of lattice ordered groups belonging to P;

$l(P)$ – the class of all lattice ordered groups H' that can be expressed as $H' = \bigcup_{i\in I} H_i$, where H_i are convex l-subgroups of H', $H_i \in P$ for each $i \in I$, and the system $\{H_i\}_{i\in I}$ (partially ordered by inclusion) is a chain.

Proposition 6. (Cf. [2], Thm. 2.9.) *Let $P \neq \emptyset$ be a class of linearly ordered groups. Then $[P] = d(l(h(S_c(P))))$.*

For any pair of ordinals δ_0, δ we denote by $s(\delta_0, \delta)$ the class of all ordinals τ such that $\tau = \delta_0 + \iota\delta$ for some $\iota(\tau) \in \mathrm{Ord}$.

Let R_0 and R_1 be the additive group of all integers or all reals, respectively, with the natural linear order. For the notion of the lexicographic product of linearly ordered groups cf. Conrad [1]. If I is a linearly ordered set and G_i is a linearly ordered group for each $i \in I$, then $\Gamma_{i\in I} G_i$ denotes the corresponding lexicographic product.

Let δ_0, δ and $\varkappa$ be ordinals, $\delta > 0$. Put

$$G(\delta_0, \delta, \varkappa) = \Gamma_{\tau<\varkappa} G_\tau ,$$

where $G_\tau = R_0$ if $\tau \in s(\delta_0, \delta)$, and $G_\tau = R_1$ otherwise. Further, let P_δ be the class of all lattice ordered groups $G(\delta_0, \delta, \varkappa)$ (where δ_0 and $\varkappa$ run over the class Ord).

If H is a convex l-subgroup of $G(\delta_0, \delta, \varkappa)$, then there is an ordinal $\varkappa_1 \leqq \varkappa$ such that $H = \Gamma_{\tau<\varkappa_1} G_\tau$. If H_1 is a homomorphic image of $G(\delta_0, \delta, \varkappa)$, then there is $\varkappa_1 \leqq \varkappa$ such that H_1 is isomorphic with $\Gamma_{\varkappa_1\leqq\tau<\varkappa} G_\tau$. Hence we have:

Lemma 1. *For each $0 < \delta \in \mathrm{Ord}$, $S_c(P_\delta) = h(P_\delta) = P_\delta$.*

Lemma 2. *For each $0 < \delta \in \mathrm{Ord}$, $l(P_\delta) = P_\delta$.*

The proof is simple.

If $\delta_1, \delta_2, \delta_0, \delta'_0 \in \mathrm{Ord}$, $0 < \delta_1 < \delta_2$, then $s(\delta_0, \delta_1) \neq s(\delta'_0, \delta_2)$. In fact, if $\tau \in s(\delta_0, \delta_1) \cap s(\delta'_0, \delta_2)$, then $\tau + \delta_1 \in s(\delta_0, \delta_1)$, but $\tau + \delta_1 \notin s(\delta'_0, \delta_2)$. Hence it follows that $P_\delta \cap P_\varepsilon = \emptyset$ whenever δ, ε are distinct ordinals. Therefore according to Proposition 6, Lemma 1 and Lemma 2 we obtain:

Lemma 3. *If δ and ε are distinct ordinals, $\delta > 0$, $\varepsilon > 0$, then $[P_\delta] \cap [P_\varepsilon] = \bar{0}$.*

Put $P = \bigcup_{0<\delta\in\mathrm{Ord}} P_\delta$, $C = [P]$.

Lemma 4. $C = d(P)$.

Proof. Lemmas 1–3 imply $S_c(P) = h(P) = l(P) = P$, hence according to Proposition 6 we have $C = d(P)$.

Corollary 2. $C = \bigvee_{0<\delta\in\mathrm{Ord}} [P_\delta]$.

For each torsion class K with $0 \neq K \leqq C$ let Ord_K be the class of all ordinals

$\delta > 0$ with $K \cap [P_\delta] \neq \bar{0}$. Corollary 2 implies (cf. also [3])

$$(1) \qquad K = K \wedge C = K \wedge (\bigvee_{0 < \delta \in \mathrm{Ord}} [P_\delta]) = \bigvee_{0 < \delta \in \mathrm{Ord}} (K \wedge [P_\delta]) = = \bigvee_{\delta \in \mathrm{Ord}_K} (K \wedge [P_\delta]) .$$

Moreover, from Lemma 3 we get

$$(2) \qquad (K \wedge [P_\delta]) \wedge (K \wedge [P_\varepsilon]) = \bar{0}$$

for each pair of distinct ordinals δ, ε.

Lemma 5. *Let K be a torsion class, $K \leqq C$. Suppose that K is β-compact. Then the class Ord_K is finite.*

Proof. Assume that the class Ord_K is infinite. Then there are elements $\delta(n) \in \mathrm{Ord}_K$, $\delta(n) < \delta(n + 1)$ $(n = 1, 2, \ldots)$. Put $K_i = [P_{\delta(i)}]$ $(i = 1, 2, \ldots)$, $K_0 = \bigvee [P_\delta]$ $(\delta \in \mathrm{Ord}_K \setminus \{\delta(1), \delta(2), \delta(3), \ldots\})$. In view of (1) we have $K = \bigvee K_i$ $(i = 0, 1, 2, \ldots)$. Let n be a positive integer. Then $\bar{0} < K_{n+1} \leqq K$ and according to (2),

$$K_{n+1} \wedge (\bigvee_{i=0,1,2,\ldots,n} K_i) = \bigvee_{i=0,1,2,\ldots,n} (K_{n+1} \wedge K_i) = \bar{0},$$

whence $K \neq \bigvee_{i=0,1,2,\ldots,n} K_i$. Thus K fails to be β-compact, which is a contradiction.

Lemma 6. *Let $I \neq \emptyset$ be a set. For each $i \in I$ let K_i be a β-compact torsion class with $K_i \leqq C$. Put $C_1 = \bigvee_{i \in I} K_i$. Then $C_1 < C$.*

Proof. Let us apply analogous denotations as above. Clearly $C_1 \leqq C$. Put $\mathrm{Ord}_1 = \bigcup_{i \in I} \mathrm{Ord}_{K_i}$. From Lemma 5 it follows that Ord_1 is a set. Hence there is $\delta \in \mathrm{Ord}$ with $\delta \notin \mathrm{Ord}_1$. Then $C_1 \wedge [P_\delta] = \bar{0}$, whence $C_1 \neq C$.

Proposition 7. *The lattice* Rad *fails to be β-algebraic.*

This is a consequence of Lemma 6. Let us remark that Corollary 1 can be obtained also from Proposition 7.

References

[1] *P. Conrad:* Lattice ordered groups, Tulane University 1970.

[2] *J. Jakubík:* Torsion radicals of lattice ordered groups. Czech. Math. J. (submitted).

[3] *J. Martinez:* Torsion theory for lattice-ordered groups. Czech. Math. J. *25* (1975), 284—299.

[4] *J. Martinez:* Is the lattice of torsion classes algebraic? Proc. Amer. Math. Soc. *63* (1977), 9—14.

Author's address: 040 01 Košice, Švermova 5, ČSSR (VŠT).