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SMOOTHNESS OF A TYPICAL CONVEX FUNCTION

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Introduction. Let \( G \subset R^m \) be a nonempty open bounded convex set. Denote by \( \mathcal{F} = \mathcal{F}(G) \) the system of all bounded convex functions on \( G \) and put \( \mathcal{F}^j = \mathcal{F} \cap C^j(G) \). The set \( \mathcal{F} \) equipped with the usual supremum metric is obviously a complete metric space.

P. M. Gruber [2] 1977 proved that a typical function from \( \mathcal{F} \) is smooth but not too smooth. More specifically, one of his results reads as follows:

Sets \( \mathcal{F} \setminus \mathcal{F}^1 \) and \( \mathcal{F}^2 \) are of the first category.

The aim of this note is to give a more detailed information concerning the “gap” between \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \). A special case of our result shows that a typical function of \( \mathcal{F} \) is of the class \( C^{1+\varepsilon} \) on no (nonempty) open subset of \( G \).

Notation. Let \( \Omega \) stand for the set of all continuous increasing functions \( \omega : [0, \infty[ \rightarrow [0, \infty[ \) such that \( \omega(0) = 0 \) and \( \omega > 0 \) on \( ]0, \infty[ \). For \( \omega_1, \omega_2 \in \Omega \), we write \( \omega_1 < \omega_2 \), if \( \omega_1(t) = o(\omega_2(t)) \), \( t \to 0+ \). A set \( \Omega' \subset \Omega \) is said to be majorized, if there is \( \omega_0 \in \Omega \) such that \( \omega < \omega_0 \), whenever \( \omega \in \Omega' \).

If \( M \subset R^m \), \( \omega \in \Omega \), then \( D_\omega(M) \) is the set of functions \( g \) defined on \( M \) and satisfying

\[
|g(x) - g(y)| \leq \omega(|x - y|), \quad x, y \in M.
\]

In what follows, \( J \) denotes the set \( \{1, 2, \ldots, m\} \) and \( \partial_j f \) means the \( j \)-th partial derivative of \( f \).

Theorem. Let \( \Omega^* \subset \Omega \) be majorized and let \( \mathcal{F}^* \) be the set of all \( f \in \mathcal{F} \) possessing the following property: There exist \( j \in J \), \( \omega \in \Omega^* \) and a nonempty open set \( G^* \subset G \) such that \( \partial_j f \in D_\omega(G^*) \). Then the set \( \mathcal{F}^* \) is of the first category.

Remark. The proof of Theorem is postponed to the end of this note.

Denote by \( \Omega_H \) the set of all functions \( \omega \) for which there are \( K > 0 \) and \( \alpha > 0 \) such that \( \omega(t) = Kt^\alpha \), \( t \in [0, \infty[ \). It is easily seen that \( \Omega_H \) is majorized. (For instance, if \( \omega_0(t) = -1/\log t \) for \( t \in ]0, 1/e[ \), \( \omega_0(0) = 0 \), \( \omega_0 = 1 \) on \( ]1/e, \infty[ \), then \( \omega_0 \in \Omega \) and \( \omega < \omega_0 \) for every \( \omega \in \Omega_H \).)
It follows from Theorem that, in an obvious sense, a typical function from $\mathcal{F}$ has nowhere Hölder continuous partial derivatives. In particular, $\mathcal{F}^2$ is of the first category.

Similar assertions could be stated for other scales of moduli of continuity. In this connection, the following notion seems to be appropriate.

A set $\Omega_1 \subset \Omega$ is said to have countable character, if there is a countable set $\Omega_2 \subset \Omega_1$ such that for every $\omega_1 \in \Omega_1$ there is $\omega_2 \in \Omega_2$ with $\omega_1 < \omega_2$. (Clearly, $\Omega_H$ has countable character.) The proof of the following assertion is left to the reader:

Every subset of $\Omega$ having countable character is majorized.

The proof of Theorem is based on two lemmas. In Lemma 2, $\mathcal{F}^1$ is considered as the subspace of the metric space $\mathcal{F}$.

**Lemma 1.** Let $j \in J$, $\omega \in \Omega$ and let $B$ be an open ball with center at $z_0 \in \mathbb{R}^n$. Suppose that $\varepsilon > 0$, $d > 0$. Then there is a convex function $\varphi$ with the following properties:

$\varphi \in C^1(\mathbb{R}^n), \quad \sup \{|\varphi(z)|; \ |z - z_0| < d\} < \varepsilon$

and there are distinct points $x_0, y_0 \in B$ such that

$\partial_j \varphi(x_0) - \partial_j \varphi(y_0) \geq 3\omega(|x_0 - y_0|)$.

**Proof.** Without loss of generality we can suppose that $B = \{z \in \mathbb{R}^n; \ |z| < 2r\}$ and $d > r > 0$. Define $\omega(t) = 0$ for $t \leq 0$ and

$h(s) = \int_0^s \omega^{1/2}(t - r) \, dt, \quad s \geq 0$.

Notice that $\alpha = h(d) > 0$ and that $h$ is a continuously differentiable convex function on $[0, \infty[$ vanishing near the origin. Consequently, the function $\varphi : x \to (\varepsilon/|x|)(h(|x|))$ is convex in $\mathbb{R}^n$, $\varphi \in C^1(\mathbb{R}^n)$ and $|\varphi(z)| < \varepsilon$ provided $|z| < d$.

Put $e_j = (0, \ldots, 1, \ldots, 0)$ (1 is on the $j$-th place) and find $t_0 \in (0, r)$ such that $\omega(t_0) < (\varepsilon/3\alpha)^2$. If $x_0 = (r + t_0) e_j$, $y_0 = re_j$, then $x_0, y_0 \in B$, $|x_0 - y_0| = t_0$ and

$\partial_j \varphi(x_0) - \partial_j \varphi(y_0) = (\varepsilon/|x|)(h'(r + t_0) - h'(r)) = (\varepsilon/|x|) \omega^{1/2}(t_0)$.

Since $(3\alpha/\varepsilon) \omega^{1/2}(t_0) \leq 1$, we conclude that

$\partial_j \varphi(x_0) - \partial_j \varphi(y_0) \geq 3\omega(t_0) = 3\omega(|x_0 - y_0|)$.

**Lemma 2.** Suppose that $\omega \in \Omega$, $j \in J$ and $B \subset G$ is an open ball. Denote by $\mathcal{A}(j, \omega, B)$ the set of all $f \in \mathcal{F}^1$ such that $\partial_j f \in \mathcal{D}_\omega(B)$. Then $\mathcal{A}(j, \omega, B)$ is nowhere dense in $\mathcal{F}^1$.

**Proof.** Let $N$ be the set of positive integers. Write $\mathcal{A} = \mathcal{A}(j, \omega, B)$ and prove first that $\mathcal{A}$ is a closed subset of $\mathcal{F}^1$.

To this end assume that the sequence $\{f_n\}$ of functions belonging to $\mathcal{A}$ converges uniformly on $G$ to a function $f \in \mathcal{F}^1$. We are going to show that $f \in \mathcal{A}$.

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Fix $x, y \in B$ and choose $\delta > 0$ such that $x + \delta e_j, y + \delta e_j \in B$. For $t \in [0, \delta[$ and $n \in N$ define
\[
\alpha_n(t) = f_n(x + te_j) - f_n(x) - (f_n(y + te_j) - f_n(y)).
\]
Notice that $\alpha_n(0) = 0$ and for $s \in ]0, \delta[$,
\[
\alpha'_n(s) = \partial_j f_n(x + se_j) - \partial_j f_n(y + se_j).
\]
Since $f_n \in \mathcal{A}$, the inequality $|\alpha'_n(s)| \leq \omega(|x - y|)$ holds whenever $s \in ]0, \delta[$. Thus we have
\[
|\alpha_n(t)| \leq t \cdot \omega(|x - y|) \quad \text{for every} \quad t \in [0, \delta[.
\]
Given $t \in ]0, \delta[$,
\[
\lim_{n \to \infty} \alpha_n(t) = f(x + te_j) - f(x) - (f(y + te_j) - f(y))
\]
do that
\[
\frac{|f(x + te_j) - f(x) - f(y + te_j) - f(y)|}{t} \leq \omega(|x - y|).
\]
Letting $t \to 0^+$, we conclude that $|\partial_j f(x) - \partial_j f(y)| \leq \omega(|x - y|)$. Consequently, $f \in \mathcal{A}$ and $\mathcal{A}$ is closed (in $\mathcal{F}^1$).

Fix now $\varepsilon > 0$ and $f \in \mathcal{A}$. To finish the proof of the lemma, it is sufficient to find a function $g \in \mathcal{F}^1$ such that $g \notin \mathcal{A}$ and the distance $d(f, g)$ of $f$ and $g$ is less than $\varepsilon$.

Let $d$ be the diameter of $G$ and $\varphi, x_0, y_0$ have the same meaning as in Lemma 1. Define $g(z) = f(z) + \varphi(z), z \in G$. Then $g \in \mathcal{F}^1$ and $d(f, g) < \varepsilon$. We have
\[
\partial_j g(x_0) - \partial_j g(y_0) = \partial_j f(x_0) - \partial_j f(y_0) + \partial_j \varphi(x_0) - \partial_j \varphi(y_0) \geq
\]
\[
\geq -\omega(|x_0 - y_0|) + 3\omega(|x_0 - y_0|) = 2\omega(|x_0 - y_0|).
\]
Consequently, $g \notin \mathcal{A}$.

The proof of the lemma is complete.

**Proof of Theorem.** Choose a countable system $\{B_i; i \in I\}$ of open balls $B_i \subset G$ such that $G = \bigcup B_i$ and for every nonempty open set $G' \subset G$ there is $B_i \subset G'$. Since $\Omega^*$ is majorized, there is $\omega^* \in \Omega$ such that $\omega < \omega^*$ whenever $\omega \in \Omega^*$. It is easily seen that for every $\omega \in \Omega^*$ there is $k \in N$ such that $\omega \leq k\omega^*$ on $[0, d]$. It follows that
\[
\mathcal{F}^* = (\mathcal{F} \setminus \mathcal{F}^1) \cup \bigcup_{i \in I} \bigcup_{j \in k} \mathcal{A}(j, k\omega^*, B_i).
\]
Gruber's result states that $\mathcal{F} \setminus \mathcal{F}^1$ is of the first category. By Lemma 2, $\mathcal{A}(j, k\omega^*, B_i)$ is nowhere dense in $\mathcal{F}^1$ and, a fortiori, in $\mathcal{F}$. We conclude that $\mathcal{F}^*$ is of the first category.

**Remark.** Various questions related to differential properties of convex functions are studied e.g. in [1]–[4] where further references can be found.
References


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