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SMOOTHNESS OF A TYPICAL CONVEX FUNCTION

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Introduction. Let $G \subset \mathbb{R}^m$ be a nonempty open bounded convex set. Denote by $\mathcal{F} = \mathcal{F}(G)$ the system of all bounded convex functions on $G$ and put $\mathcal{F}^1 = \mathcal{F} \cap C^1(G)$. The set $\mathcal{F}$ equipped with the usual supremum metric is obviously a complete metric space.

P. M. Gruber [2] 1977 proved that a typical function from $\mathcal{F}$ is smooth but not too smooth. More specifically, one of his results reads as follows:

Sets $\mathcal{F} \setminus \mathcal{F}^1$ and $\mathcal{F}^2$ are of the first category.

The aim of this note is to give a more detailed information concerning the "gap" between $\mathcal{F}^1$ and $\mathcal{F}^2$. A special case of our result shows that a typical function of $\mathcal{F}$ is of the class $C^{1+\varepsilon}$ on no (nonempty) open subset of $G$.

Notation. Let $\Omega$ stand for the set of all continuous increasing functions $\omega : [0, \infty[ \rightarrow [0, \infty[$ such that $\omega(0) = 0$ and $\omega > 0$ on $]0, \infty[$. For $\omega_1, \omega_2 \in \Omega$, we write $\omega_1 \prec \omega_2$, if $\omega_1(t) = O(\omega_2(t))$, $t \rightarrow 0+$. A set $\Omega' \subset \Omega$ is said to be majorized, if there is $\omega_0 \in \Omega$ such that $\omega \prec \omega_0$, whenever $\omega \in \Omega'$.

If $M \subset \mathbb{R}^m$, $\omega \in \Omega$, then $\mathcal{D}_\omega(M)$ is the set of functions $g$ defined on $M$ and satisfying

$$|g(x) - g(y)| \leq \omega(|x - y|), \quad x, y \in M.$$ 

In what follows, $J$ denotes the set $\{1, 2, \ldots, m\}$ and $\partial_j f$ means the $j$-th partial derivative of $f$.

Theorem. Let $\Omega^* \subset \Omega$ be majorized and let $\mathcal{F}^*$ be the set of all $f \in \mathcal{F}$ possessing the following property: There exist $j \in J$, $\omega \in \Omega^*$ and a nonempty open set $G^* \subset G$ such that $\partial_j f \in \mathcal{D}_\omega(G^*)$. Then the set $\mathcal{F}^*$ is of the first category.

Remark. The proof of Theorem is postponed to the end of this note.

Denote by $\Omega_H$ the set of all functions $\omega$ for which there are $K > 0$ and $\alpha > 0$ such that $\omega(t) = Kt^\alpha$, $t \in [0, \infty[$. It is easily seen that $\Omega_H$ is majorized. (For instance, if $\omega_0(t) = -1/\log t$ for $t \in [0, 1/\varepsilon[$, $\omega_0(0) = 0$, $\omega_0 = 1$ on $[1/\varepsilon, \infty[$, then $\omega_0 \in \Omega$ and $\omega \prec \omega_0$ for every $\omega \in \Omega_H$.)

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It follows from Theorem that, in an obvious sense, a typical function from $\mathcal{F}$ has nowhere Hölder continuous partial derivatives. In particular, $\mathcal{F}^2$ is of the first category.

Similar assertions could be stated for other scales of moduli of continuity. In this connection, the following notion seems to be appropriate.

A set $\Omega_1 \subset \Omega$ is said to have countable character, if there is a countable set $\Omega_2 \subset \Omega_1$ such that for every $\omega_1 \in \Omega_1$ there is $\omega_2 \in \Omega_2$ with $\omega_1 < \omega_2$. (Clearly, $\Omega_\mathcal{H}$ has countable character.) The proof of the following assertion is left to the reader:

Every subset of $\Omega$ having countable character is majorized.

The proof of Theorem is based on two lemmas. In Lemma 2, $\mathcal{F}^1$ is considered as the subspace of the metric space $\mathcal{F}$.

**Lemma 1.** Let $j \in J$, $\omega \in \Omega$ and let $B$ be an open ball with center at $z_0 \in \mathbb{R}^m$. Suppose that $\varepsilon > 0$, $d > 0$. Then there is a convex function $\varphi$ with the following properties:

$$\varphi \in C^1(\mathbb{R}^m), \quad \sup \{|\varphi(z)|; \ |z - z_0| < d\} < \varepsilon$$

and there are distinct points $x_0, y_0 \in B$ such that

$$\partial_j \varphi(x_0) - \partial_j \varphi(y_0) \geq 3\omega(|x_0 - y_0|).$$

**Proof.** Without loss of generality we can suppose that $B = \{z \in \mathbb{R}^m; \ |z| < 2r\}$ and $d > r > 0$. Define $\omega(t) = 0$ for $t \leq 0$ and

$$h(s) = \int_0^s \omega^{1/2}(t - r) \, dt, \quad s \geq 0.$$  

Notice that $a = h(d) > 0$ and that $h$ is a continuously differentiable convex function on $[0, \infty[$ vanishing near the origin. Consequently, the function $\varphi : x \rightarrow (\varepsilon/|x|)(h(|x|))$ is convex in $\mathbb{R}^m$, $\varphi \in C^1(\mathbb{R}^m)$ and $|\varphi(z)| < \varepsilon$ provided $|z| < d$.

Put $e_j = (0, \ldots, 1, \ldots, 0)$ (1 is on the $j$-th place) and find $t_0 \in (0, r)$ such that $\omega(t_0) < (\varepsilon/3\alpha)^2$. If $x_0 = (r + t_0)e_j, \ y_0 = re_j$, then $x_0, y_0 \in B, \ |x_0 - y_0| = t_0$ and

$$\partial_j \varphi(x_0) - \partial_j \varphi(y_0) = (\varepsilon/\alpha)(h'(r + t_0) - h'(r)) = (\varepsilon/\alpha) \omega^{1/2}(t_0).$$

Since $(3\alpha/\varepsilon) \omega^{1/2}(t_0) \leq 1$, we conclude that

$$\partial_j \varphi(x_0) - \partial_j \varphi(y_0) \geq 3\omega(t_0) = 3\omega(|x_0 - y_0|).$$

**Lemma 2.** Suppose that $\omega \in \Omega$, $j \in J$ and $B \subset G$ is an open ball. Denote by $\mathcal{A}(j, \omega, B)$ the set of all $f \in \mathcal{F}^1$ such that $\partial_j f \in D_\omega(B)$. Then $\mathcal{A}(j, \omega, B)$ is nowhere dense in $\mathcal{F}^1$.

**Proof.** Let $N$ be the set of positive integers. Write $\mathcal{A} = \mathcal{A}(j, \omega, B)$ and prove first that $\mathcal{A}$ is a closed subset of $\mathcal{F}^1$.

To this end assume that the sequence $\{f_n\}$ of functions belonging to $\mathcal{A}$ converges uniformly on $G$ to a function $f \in \mathcal{F}^1$. We are going to show that $f \in \mathcal{A}$.
Fix $x, y \in B$ and choose $\delta > 0$ such that $x + \delta e_j, y + \delta e_j \in B$. For $t \in [0, \delta[$ and $n \in N$ define
\[
\alpha_n(t) = f_n(x + te_j) - f_n(x) - (f_n(y + te_j) - f_n(y)).
\]
Notice that $\alpha_n(0) = 0$ and for $s \in ]0, \delta[$,
\[
\alpha'_n(s) = \partial_j f_n(x + se_j) - \partial_j f_n(y + se_j).
\]
Since $f_n \in \mathcal{A}$, the inequality $|\alpha'_n(s)| \leq \omega(|x - y|)$ holds whenever $s \in ]0, \delta[$. Thus we have
\[
|\alpha_n(t)| \leq t \cdot \omega(|x - y|) \quad \text{for every} \quad t \in [0, \delta[.
\]
Given $t \in ]0, \delta[$,
\[
\lim_{n \to \infty} \alpha_n(t) = f(x + te_j) - f(x) - (f(y + te_j) - f(y))
\]
so that
\[
\left| \frac{f(x + te_j) - f(x)}{t} - \frac{f(y + te_j) - f(y)}{t} \right| \leq \omega(|x - y|).
\]
Letting $t \to 0^+$, we conclude that $|\partial_j f(x) - \partial_j f(y)| \leq \omega(|x - y|)$. Consequently, $f \in \mathcal{A}$ and $\mathcal{A}$ is closed (in $\mathcal{F}^1$).

Fix now $\varepsilon > 0$ and $f \in \mathcal{A}$. To finish the proof of the lemma, it is sufficient to find a function $g \in \mathcal{F}^1$ such that $g \notin \mathcal{A}$ and the distance $\varrho(f, g)$ of $f$ and $g$ is less than $\varepsilon$.

Let $d$ be the diameter of $G$ and $\varphi, x_0, y_0$ have the same meaning as in Lemma 1. Define $g(z) = f(z) + \varphi(z), z \in G$. Then $g \in \mathcal{F}^1$ and $\varrho(f, g) < \varepsilon$. We have
\[
\partial_j g(x_0) - \partial_j g(y_0) = \partial_j f(x_0) - \partial_j f(y_0) + \partial_j \varphi(x_0) - \partial_j \varphi(y_0) \geq
\]
\[
\geq -\omega(|x_0 - y_0|) + 3\omega(|x_0 - y_0|) = 2\omega(|x_0 - y_0|).
\]
Consequently, $g \notin \mathcal{A}$.

The proof of the lemma is complete.

**Proof of Theorem.** Choose a countable system $\{B_i; i \in I\}$ of open balls $B_i \in G$ such that $G = \bigcup B_i$ and for every nonempty open set $G' \subset G$ there is $B_i \subset G'$. Since $\Omega^*$ is majorized, there is $\omega^* \in \Omega$ such that $\omega < \omega^*$ whenever $\omega \in \Omega^*$. It is easily seen that for every $\omega \in \Omega^*$ there is $k \in N$ such that $\omega \leq k\omega^*$ on $[0, d]$. It follows that
\[
\mathcal{F}^* \subset (\mathcal{F} \setminus \mathcal{F}^1) \cup \bigcup_{i \in I} \bigcup_{j \in \mathcal{A}(j, k\omega^*, B_i)}.
\]
Gruber's result states that $\mathcal{F} \setminus \mathcal{F}^1$ is of the first category. By Lemma 2, $\mathcal{A}(j, k\omega^*, B_i)$ is nowhere dense in $\mathcal{F}^1$ and, a fortiori, in $\mathcal{F}$. We conclude that $\mathcal{F}^*$ is of the first category.

**Remark.** Various questions related to differential properties of convex functions are studied e.g. in [1]–[4] where further references can be found.
References


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