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DIRECT DECOMPOSABILITY OF TOLERANCES ON LATTICES,  
SEMILATTICES AND QUASILATTICES

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In the paper [2] the authors considered tolerances on direct products of monoids and distributive lattices in order to obtain conditions under which a tolerance is a direct product of tolerances on direct factors. Actually, the following result is proved:

**Theorem.** *Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two monoids or two distributive lattices with greatest and least elements. Then the following two implications are equivalent:*

- (1)  $T \in LT(\mathfrak{A} \times \mathfrak{B}) \Rightarrow$  there exist  $T_1 \in LT(\mathfrak{A}), T_2 \in LT(\mathfrak{B})$  such that  $T = T_1 \times T_2$ .
- (2)  $\langle a, b \rangle \in T \Rightarrow T_A(pr_1a, pr_1b) \times T_B(pr_2a, pr_2b) \subseteq T$ .

The aim of this paper is twofold:

- to extend the above result for algebras in the title;
- to prove that (2) of Theorem holds automatically in lattices, i.e. lattices have directly decomposable tolerances without any constraints.

0. BASIC CONCEPTS

Let  $\mathfrak{A} = (A, F)$  be an algebra. By a *tolerance*  $T$  (or *tolerance relation*) on  $\mathfrak{A}$  we mean a reflexive and symmetric binary relation on  $A$  with the *Substitution Property* with respect to  $F$ , i.e.  $T$  is a subalgebra of the direct product  $\mathfrak{A} \times \mathfrak{A}$ . The set of all tolerances on an algebra  $\mathfrak{A}$  constitutes an algebraic lattice  $LT(\mathfrak{A})$  [1], and the meet in  $LT(\mathfrak{A})$  coincides with the set intersection. We denote the join in  $LT(\mathfrak{A})$  by  $\vee_A$ .

Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two algebras of the same type,  $\mathfrak{A} \times \mathfrak{B}$  their direct product and  $T \in LT(\mathfrak{A} \times \mathfrak{B})$ .  $T$  is called *directly decomposable* if there exist  $T_1 \in LT(\mathfrak{A})$  and  $T_2 \in LT(\mathfrak{B})$  such that  $T = T_1 \times T_2$ . If every tolerance on  $\mathfrak{A} \times \mathfrak{B}$  is directly decomposable, we say that  $\mathfrak{A} \times \mathfrak{B}$  has directly decomposable tolerances. If  $\mathcal{C}$  is a class of algebras such that for every pair  $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ ,  $\mathfrak{A} \times \mathfrak{B}$  has directly decomposable tolerances,  $\mathcal{C}$  is said to have *directly decomposable tolerances*.

Let  $a$  and  $b$  be two elements of an algebra  $\mathfrak{A}$ .  $T_A(a, b)$  denotes the *least tolerance*

on  $\mathfrak{A}$  collapsing the pair  $\langle a, b \rangle$ , i.e.  $T_A(a, b) = \bigcap \{T \mid T \in LT(\mathfrak{A}) \text{ and } \langle a, b \rangle \in T\}$ . Thus  $T_A(a, b)$  is a generalization of the concept of a principal congruence.

Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two algebras of the same type and  $x$  an element of  $\mathfrak{A} \times \mathfrak{B}$ . When  $x_1 = pr_1x$  and  $x_2 = pr_2x$ ,  $[x_1, x_2]$  is a componentwise denotation for  $x$ . Further, if  $T_1 \in LT(\mathfrak{A})$  and  $T_2 \in LT(\mathfrak{B})$ , we have  $\langle x, y \rangle \in T_1 \times T_2$  if  $\langle x_1, y_1 \rangle \in T_1$  and  $\langle x_2, y_2 \rangle \in T_2$ . As noted in [2], the direct product of two tolerances is a tolerance on the direct product of the corresponding algebras.

## 1. TOLERANCES ON DIRECT PRODUCTS

The aim of this section is to give conditions under which the identity

$$(1) \quad (T_1 \times T_2) \vee_{A \times B} (S_1 \times S_2) = (T_1 \vee_A S_1) \times (T_2 \vee_B S_2)$$

is valid for two algebras  $\mathfrak{A}$  and  $\mathfrak{B}$  of the same type and for every  $T_1, S_1 \in LT(\mathfrak{A})$  and every  $T_2, S_2 \in LT(\mathfrak{B})$ . It is worth noting that (1) holds for congruences  $T_1, T_2, S_1$  and  $S_2$  on any algebras  $\mathfrak{A}$  and  $\mathfrak{B}$  of the same type, see [3].

An algebra is called *idempotent* if for every  $m$ -ary polynomial  $q(x_1, \dots, x_m)$  over  $\mathfrak{A}$  and for every  $a \in A$ ,  $q(a, \dots, a) = a$ .  $\mathfrak{A}$  is called *superidempotent* if it is idempotent and for every  $m$ -ary polynomial  $q$  and every two elements  $a$  and  $b$  of  $\mathfrak{A}$  there are elements  $c$  and  $d$  such that  $q(k, \dots, k, a, k, \dots, k) = a$  and  $q(k, \dots, k, b, k, \dots, k) = b$ , where  $k$  is  $c$  or  $d$  according to the following rule: if  $k$  on  $i$ th place is  $c$  ( $d$ ) in the expression for  $a$  then  $k$  on the  $i$ th place is also  $c$  ( $d$ ) in the expression for  $b$ , and vice versa. When  $\mathfrak{A}$  is a lattice, it is superidempotent: the elements  $c$  and  $d$  corresponding to given  $a$  and  $b$  are  $a \vee b$  and  $a \wedge b$ .

**Theorem 1.** *Let  $\mathcal{C}$  be a class of superidempotent algebras of the same type, where for every  $\mathfrak{A} \in \mathcal{C}$  and every  $T, S \in LT(\mathfrak{A})$ ,  $\langle u, v \rangle \in T \vee_A S$  if and only if there exists a positive integer  $N$  such that for every even  $n > N$  there are elements  $u_i, v_i$  of  $\mathfrak{A}$  ( $i = 1, \dots, n$ ) and an  $n$ -ary polynomial  $p$  over  $\mathfrak{A}$  with the properties*

- (i)  $p(u_1, \dots, u_n) = u$  and  $p(v_1, \dots, v_n) = v$ ;
- (ii)  $\langle u_i, v_i \rangle \in T$  for even and  $\langle u_i, v_i \rangle \in S$  for odd values of  $i$  ( $i = 1, \dots, n$ ).

*Then the identity (1) is valid for all  $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$  and every  $T_1, S_1 \in LT(\mathfrak{A})$ ,  $T_2, S_2 \in LT(\mathfrak{B})$ .*

*Proof.* The proof is a modification of the proof of [2, Thm. 1]. Evidently  $T_1 \times T_2, S_1 \times S_2 \subseteq (T_1 \vee_A S_1) \times (T_2 \vee_B S_2)$ , whence it remains to prove the inclusion  $(T_1 \vee_A S_1) \times (T_2 \vee_B S_2) \subseteq (T_1 \times T_2) \vee_{A \times B} (S_1 \times S_2)$ .

Let  $a_1, b_1 \in A$  and  $a_2, b_2 \in B$  be elements such that  $\langle [a_1, a_2], [b_1, b_2] \rangle \in (T_1 \vee_A S_1) \times (T_2 \vee_B S_2)$ . Then  $\langle a_1, b_1 \rangle \in T_1 \vee_A S_1$  and  $\langle a_2, b_2 \rangle \in T_2 \vee_B S_2$ . According to the assumption, there are two positive integers  $N_1$  and  $N_2$  such that for every even integer  $n > \max(N_1, N_2)$  there exist elements  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$

of  $\mathfrak{A}$ ,  $u'_1, \dots, u'_n$  and  $v'_1, \dots, v'_n$  of  $\mathfrak{B}$ , and  $n$ -ary polynomials  $p$  and  $q$  over  $\mathfrak{A}$  and  $\mathfrak{B}$  having the properties

- (i)  $p(u_1, \dots, u_n) = a_1$ ,  $p(v_1, \dots, v_n) = b_1$ ,  $q(u'_1, \dots, u'_n) = a_2$  and  $q(v'_1, \dots, v'_n) = b_2$ ;
- (ii)  $\langle u_i, v_i \rangle \in T_1$  and  $\langle u'_i, v'_i \rangle \in T_2$  for even and  $\langle u_i, v_i \rangle \in S_1$  and  $\langle u'_i, v'_i \rangle \in S_2$  for odd values of  $i$ ,  $i = 1, \dots, n$ .

We define now an  $n^2$ -ary polynomial  $r$  as follows:  $r(y_1, \dots, y_{n^2}) = p(q(x_{11}, \dots, x_{1n}), \dots, q(x_{n1}, \dots, x_{nn}))$ , where  $x_{k,j} = y_{(k-1)n+j}$ . Let  $c_i$  and  $d_i$  be elements of  $\mathfrak{A}$  such that  $q(k_i, \dots, k_i, u_i, k_i, \dots, k_i) = u_i$  and  $q(k_i, \dots, k_i, v_i, k_i, \dots, k_i) = v_i$ , where  $k_i$  is  $c_i$  or  $d_i$ ,  $i = 1, \dots, n$ . Furthermore,  $\langle c_i, c_i \rangle, \langle d_i, d_i \rangle \in S_1, T_1$  for every value of  $i$ . Accordingly, we can now write the following scheme:

- $t$  and  $s$  are indices,  $t, s = 1, \dots, n$ ;
- for each separate value of  $t$ ,  $z_s = w_s = k_t$  when  $s \neq t$ , and  $z_s = u_t$ ,  $w_s = v_t$  when  $s = t$ ;
- $\langle [z_s, u'_t], [w_s, v'_t] \rangle \in T_1 \times T_2$  when  $t$  is even and  $\langle [z_s, u'_t], [w_s, v'_t] \rangle \in S_1 \times S_2$  when  $t$  is odd.

Moreover,  $r([z_1, u'_1], [z_2, u'_1], \dots, [z_n, u'_n]) = [r(z_1, \dots, z_n, z_1, \dots, z_n, \dots, z_n)$ ,  $r(u'_1, \dots, u'_1, u'_2, \dots, u'_2, u'_3, \dots, u'_n)] = [r(u_1, k_1, \dots, k_1, k_2, u_2, k_2, \dots, k_2, k_3, \dots, \dots, k_n, u_n)$ ,  $r(u'_1, \dots, u'_1, u'_2, \dots, u'_n)] = [a_1, a_2]$  and similarly  $r([w_1, v'_1], [w_2, v'_1], \dots, [w_n, v'_n]) = [b_1, b_2]$ . Obviously,  $\mathfrak{A} \times \mathfrak{B}$  is superidempotent when  $\mathfrak{A}$  and  $\mathfrak{B}$  are, whence one can derive from the polynomial  $r$  a new polynomial  $r^*$  such that  $r^*$  is  $(n^2 + j)$ -ary, where  $j$  is even and  $r^*(y_1, \dots, y_{n^2+j}) = pr_1(r(y_1, \dots, y_{n^2}), y_{n^2+1}, \dots, y_{n^2+j})$ . But then, according to the superidempotency of  $\mathfrak{A} \times \mathfrak{B}$  with respect to  $r^*$ ,  $[a_1, a_2]$  and  $[b_1, b_2]$ , we have two sequences of elements and a polynomial  $r^*$  over  $\mathfrak{A} \times \mathfrak{B}$  such that (i) and (ii) hold for every even  $m > n^2 - 2$  and thus  $\langle [a_1, a_2], [b_1, b_2] \rangle \in (T_1 \times T_2) \vee_{A \times B} (S_1 \times S_2)$ . This completes the proof.

A *join-semilattice*  $\mathfrak{S} = (S, \vee)$  is called *down directed*, if for any two elements  $a, b \in S$  there is a common lower bound  $c$  of  $a$  and  $b$  in  $\mathfrak{S}$ . An *up directed meet-semilattice* is defined dually. A *quasilattice*  $\mathfrak{Q} = (Q, \vee, \wedge)$  is a structure, where  $\vee$  and  $\wedge$  are commutative, associative and idempotent (see Płonka [4]), i.e.  $\mathfrak{Q}$  is a join-semilattice with respect to  $\vee$  and a meet-semilattice with respect to  $\wedge$ .  $\mathfrak{Q}$  is a lattice if and only if the absorption laws hold in  $\mathfrak{Q}$ .  $\mathfrak{Q}$  is down directed, if it is down directed as a join-semilattice, and up directed, if it is up directed as a meet-semilattice. Obviously, down directed join-semilattices and quasilattices as well as up directed meet-semilattices and quasilattices are superidempotent.

**Theorem 2.** *Let  $\mathcal{C}$  be one of the following classes of algebras:*

- (i) *the class of all lattices;*
- (ii) *the class of all down directed join-semilattices;*
- (iii) *the class of all up directed meet-semilattices;*

(iv) the class of all down directed quasilattices;

(v) the class of all up directed quasilattices.

Then (1) is true for each  $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$  and for every  $T_1, S_1 \in LT(\mathfrak{A}), T_2, S_2 \in LT(\mathfrak{B})$ .

Proof. We have to show that when  $\mathfrak{A} \in \mathcal{C}, T, S \in LT(\mathfrak{A})$  and  $u, v \in A$ , then  $\langle u, v \rangle \in T \vee_A S$  if and only if there exists an even positive integer  $N$  such that for every even integer  $n > N$  there are elements  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  and an  $n$ -ary polynomial  $p$  over  $\mathfrak{A}$  such that (i) and (ii) of Theorem 1 hold. After proving this the assertion of the theorem follows from Theorem 1. We shall present the proof only for lattices; the proofs for (ii)–(v) are analogous and hence we omit them.

As proved in [1, Thm. 2],  $\langle u, v \rangle \in T \vee_A S$  if and only if there is a polynomial  $p^*(y_1, \dots, y_m)$  and elements  $u_1^*, \dots, u_m^*$  and  $v_1^*, \dots, v_m^*$  such that  $\langle u_i^*, v_i^* \rangle \in T$  or  $\langle u_i^*, v_i^* \rangle \in S, i = 1, \dots, m, p^*(u_1^*, \dots, u_m^*) = u$  and  $p^*(v_1^*, \dots, v_m^*) = v$ . Thus if the conditions (i) and (ii) of Theorem 1 hold, then  $\langle u, v \rangle \in T \vee_A S$ . So it remains to show the converse and we shall do it by modifying the polynomial  $p^*(y_1, \dots, y_m)$  and the sequences  $u_1^*, \dots, u_m^*$  and  $v_1^*, \dots, v_m^*$  in a suitable manner.

Let us denote  $u_1^* \wedge \dots \wedge u_m^* \wedge v_1^* \wedge \dots \wedge v_m^*$  by  $a^*$ . Trivially,  $\langle a^*, a^* \rangle \in T, S, u_i^* \vee a^* = u_i^*$  and  $v_i^* \vee a^* = v_i^*$  for each  $i, i = 1, \dots, m$ . If  $\langle u_1^*, v_1^* \rangle \in S$ , we put  $u_1 = u_1^*$  and  $v_1 = v_1^*$ , and if  $\langle u_1^*, v_1^* \rangle \notin S$ , we put  $u_1 = v_1 = a^*, u_2 = u_1^*$  and  $v_2 = v_1^*$ , clearly then  $\langle u_2, v_2 \rangle \in T$ . Assume that  $\langle u_1^*, v_1^* \rangle \in S$ , whence  $u_1 = u_1^*$  and  $v_1 = v_1^*$ . If now  $\langle u_2^*, v_2^* \rangle \in T$ , we put  $u_2 = u_2^*$  and  $v_2 = v_2^*$ , and if  $\langle u_2^*, v_2^* \rangle \notin T$ , then we put  $u_2 = v_2 = a^*$ . In that case  $\langle u_2, v_2 \rangle \in T$  and because then  $\langle u_2^*, v_2^* \rangle \notin S$ , we put  $u_2^* = u_3$  and  $v_2^* = v_3$ . So from  $u_1^*, \dots, u_m^*$ , from  $v_1^*, \dots, v_m^*$  and from  $a^*$  we can easily construct two new sequences  $u_1, \dots, u_{2k}$  and  $v_1, \dots, v_{2k}$  such that  $\langle u_i, v_i \rangle \in T$  for even and  $\langle u_i, v_i \rangle \in S$  for odd values of  $i, i = 1, \dots, 2k$ . Assume that  $\langle u_1^*, v_1^* \rangle \notin S$ , and so  $u_1 = v_1 = a^*, u_2 = u_1^*$  and  $v_2 = v_1^*$ . Then we replace  $y_1$  in the polynomial  $p^*(y_1, \dots, y_m)$  by the expression  $x_1 \vee x_2$  and obtain a new polynomial  $p'(x_1, x_2, y_2, \dots, y_m)$ . After performing all similar necessary modifications in the polynomial  $p^*$  we have a new one:  $p(x_1, \dots, x_{2k})$ . Because  $u_i^* \vee a^* = u_i^*$  and  $v_i^* \vee a^* = v_i^*$ ,  $p(u_1, \dots, u_{2k}) = u$  and  $p(v_1, \dots, v_{2k}) = v$ . Now we may put  $N = 2k - 2$  and if  $n > N$ , we put  $u_i = v_i = a^*$  for  $i = 2k + 1, \dots, n$ , and moreover  $p(x_1, \dots, x_n) = p(x_1, \dots, x_{2k}) \vee x_{2k+1} \vee \dots \vee x_n$ . In this case the conditions (i) and (ii) of Theorem 1 also hold, and the required result follows from [1, Thm. 2].

It is proved in [2] that the identity (1) implies a similar identity for an arbitrary number of tolerances on direct factors, i.e.

$$(2) \quad \mathbf{V}_{A \times B} \{T_\gamma \times S_\gamma \mid \gamma \in \Gamma\} = \mathbf{V}_A \{T_\gamma \mid \gamma \in \Gamma\} \times \mathbf{V}_B \{S_\gamma \mid \gamma \in \Gamma\}$$

( $\Gamma$  is an arbitrary index set) in the class of all distributive lattices with a least and a greatest element as well as in the class of all monoids with a unit element. In the following we extend this result. The proof follows from that of [2, Thm. 2], where the unit element is substituted by a lower bound (by an upper bound) of the elements under consideration and the operation  $\circ$  by  $\vee$  (by  $\wedge$ ). Hence the proof is omitted.

**Theorem 3.** Let  $\mathcal{C}$  be one of the classes (i)–(v) of algebras in Theorem 2. Then (2) is valid for each pair  $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$  and for every  $T_\gamma \in LT(\mathfrak{A}), S_\gamma \in LT(\mathfrak{B})$ , where  $\Gamma$  is an arbitrary index set.

## 2. DIRECT DECOMPOSABILITY

Theorem 5 of [2] can be generalized in the following way:

**Theorem 4.** Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two algebras of the same type satisfying (2). Then the following conditions are equivalent:

- (1)  $\mathfrak{A} \times \mathfrak{B}$  has directly decomposable tolerances;
- (2)  $\langle a, b \rangle \in T$  implies  $T_A(a_1, b_1) \times T_B(a_2, b_2) \subseteq T$  for each  $T \in LT(\mathfrak{A})$ .

*Proof.* (1)  $\Rightarrow$  (2). The equality  $T_{A \times B}(a, b) = T_1 \times T_2$  evidently implies that  $T_A(a_1, b_1) \subseteq T_1, T_B(a_2, b_2) \subseteq T_2$ , and thus  $\langle a, b \rangle \in T$  implies that  $T_A(a_1, b_1) \times T_B(a_2, b_2) \subseteq T_1 \times T_2 \subseteq T$ .

(2)  $\Rightarrow$  (1). Let

$T_1 = \{ \langle a_1, b_1 \rangle \mid \text{there exist } a_2, b_2 \text{ of } \mathfrak{B} \text{ such that}$

$$\langle [a_1, a_2], [b_1, b_2] \rangle \in T \}$$
 and

$T_2 = \{ \langle a_2, b_2 \rangle \mid \text{there exist } a_1, b_1 \text{ of } \mathfrak{A} \text{ such that}$

$$\langle [a_1, a_2], [b_1, b_2] \rangle \in T \}.$$

By Theorem 14 in [1],  $T_1 = \bigvee_A \{ T_A(a_1, b_1) \mid \langle a, b \rangle \in T \}$  and  $T_2 = \bigvee_B \{ T_B(a_2, b_2) \mid \langle a, b \rangle \in T \}$ . Then it follows from (2) that  $T_1 \times T_2 = (\bigvee_A \{ T_A(a_1, b_1) \mid \langle a, b \rangle \in T \}) \times (\bigvee_B \{ T_B(a_2, b_2) \mid \langle a, b \rangle \in T \}) \subseteq \bigvee_{A \times B} \{ T_A(a_1, b_1) \times T_B(a_2, b_2) \mid \langle a, b \rangle \in T \} \subseteq T$ . The converse inclusion is evident. Because  $T_1 \in LT(\mathfrak{A})$  and  $T_2 \in LT(\mathfrak{B})$ , (1) is proved.

Next we shall prove two lemmas, by means of which we can prove the second result from the introduction.

**Lemma 1.** Let  $\mathfrak{A} = (A, F)$  be an algebra and  $a, b \in A$ .  $\langle x, y \rangle \in T_A(a, b)$  if and only if there exists a binary algebraic function  $\varphi$  over  $\mathfrak{A}$  such that  $x = \varphi(a, b)$  and  $y = \varphi(b, a)$ .

*Proof.* Clearly the set of all pairs  $\langle x, y \rangle$  for all binary algebraic functions  $\varphi$  from the theorem over  $\mathfrak{A}$  constitute a reflexive and symmetric binary relation  $T$  having the Substitution Property and collapsing  $\langle a, b \rangle$ , i.e.  $T_A(a, b) \subseteq T$ . The converse inclusion is evident.

**Lemma 2.** Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two lattices. Then  $T_A(a_1, b_1) \times T_B(a_2, b_2) \subseteq T_{A \times B}(a, b)$  for every pair  $\langle a, b \rangle$  of elements of  $\mathfrak{A} \times \mathfrak{B}$ .

**Proof.** Let  $\langle x, y \rangle \in T_A(a_1, b_1) \times T_B(a_2, b_2)$ . Then  $\langle x_1, y_1 \rangle \in T_A(a_1, b_1)$ ,  $\langle x_2, y_2 \rangle \in T_B(a_2, b_2)$  and, according to Lemma 1, there exist  $(2 + n)$ -ary and  $(2 + m)$ -ary polynomials  $p$  and  $q$  such that  $x_1 = p(a_1, b_1, c_1, \dots, c_n)$ ,  $y_1 = p(b_1, a_1, c_1, \dots, c_n)$ ,  $x_2 = q(a_2, b_2, d_1, \dots, d_m)$  and  $y_2 = q(b_2, a_2, d_1, d_2, \dots, d_m)$ . Let  $s = \max(m, n)$  and let us put  $c_i = c_n$  and  $d_j = d_m$  for  $i = n, \dots, s$  and  $j = m, \dots, s$ . Now we can construct a  $(4 + s)$ -ary polynomial  $r$  as follows:  $r(x, y, k_1, \dots, k_s, e_1, e_2) = (e_1 \wedge p(x, y, k_1, \dots, k_s)) \vee (e_2 \wedge q(x, y, k_1, \dots, k_s))$ . But then  $p(x, y, c_1, \dots, c_n) = r(x, y, c_1, \dots, c_s, h, g)$  and  $q(x, y, d_1, \dots, d_m) = r(x, y, d_1, \dots, d_s, g, h)$ , where  $h = x \vee y \vee c_1 \vee \dots \vee c_n \vee d_1 \vee \dots \vee d_m$  and  $g = x \wedge y \wedge c_1 \wedge \dots \wedge c_n \wedge d_1 \wedge \dots \wedge d_m$ . Further,  $\langle x, y \rangle = \langle [x_1, x_2], [y_1, y_2] \rangle = \langle [r(a_1, b_1, c_1, \dots, c_s, h, g), r(a_2, b_2, d_1, \dots, d_s, g, h)], [r(b_1, a_1, c_1, \dots, c_s, h, g), r(b_2, a_2, d_1, \dots, d_s, g, h)] \rangle = \langle r(a, b, [c_1, d_1], \dots, [c_s, d_s], [h, g], [g, h]), r(a, b, [c_1, d_1], \dots, [c_s, d_s], [h, g], [g, h]) \rangle = \langle \varphi(a, b), \varphi(b, a) \rangle$ , where  $\varphi(x, y) = r(x, y, [c_1, d_1], \dots, [c_s, d_s], [h, g], [g, h])$ . According to Lemma 1,  $\langle x, y \rangle \in T_{A \times B}(a, b)$ . This completes the proof.

Now we can prove

**Theorem 5.** *The class of all lattices has directly decomposable tolerances.*

**Proof.** By Theorem 3, the class from the theorem satisfies the identity (2), and thus Theorem 4 can be used. According to Lemma 2, 2) of Theorem 4 holds, whence the proof is a direct consequence of Theorem 4.

#### References

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