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DIRECT DECOMPOSABILITY OF TOLERANCES ON LATTICES, SEMILATTICES AND QUASILATTICES

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In the paper [2] the authors considered tolerances on direct products of monoids and distributive lattices in order to obtain conditions under which a tolerance is a direct product of tolerances on direct factors. Actually, the following result is proved:

Theorem. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be two monoids or two distributive lattices with greatest and least elements. Then the following two implications are equivalent:

1. \( T \in LT(\mathfrak{A} \times \mathfrak{B}) \Rightarrow \) there exist \( T_1 \in LT(\mathfrak{A}), T_2 \in LT(\mathfrak{B}) \) such that \( T = T_1 \times T_2 \).
2. \( \langle a, b \rangle \in T \Rightarrow T_1(pr_1a, pr_1b) \times T_2(pr_2a, pr_2b) \subseteq T \).

The aim of this paper is twofold:

- to extend the above result for algebras in the title;
- to prove that (2) of Theorem holds automatically in lattices, i.e. lattices have directly decomposable tolerances without any constraints.

0. BASIC CONCEPTS

Let \( \mathfrak{A} = (A, F) \) be an algebra. By a tolerance \( T \) (or tolerance relation) on \( \mathfrak{A} \) we mean a reflexive and symmetric binary relation on \( A \) with the Substitution Property with respect to \( F \), i.e. \( T \) is a subalgebra of the direct product \( \mathfrak{A} \times \mathfrak{A} \). The set of all tolerances on an algebra \( \mathfrak{A} \) constitutes an algebraic lattice \( LT(\mathfrak{A}) \) [1], and the meet in \( LT(\mathfrak{A}) \) coincides with the set intersection. We denote the join in \( LT(\mathfrak{A}) \) by \( \vee_A \).

Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be two algebras of the same type, \( \mathfrak{A} \times \mathfrak{B} \) their direct product and \( T \in LT(\mathfrak{A} \times \mathfrak{B}) \). \( T \) is called directly decomposable if there exist \( T_1 \in LT(\mathfrak{A}) \) and \( T_2 \in LT(\mathfrak{B}) \) such that \( T = T_1 \times T_2 \). If every tolerance on \( \mathfrak{A} \times \mathfrak{B} \) is directly decomposable, we say that \( \mathfrak{A} \times \mathfrak{B} \) has directly decomposable tolerances. If \( \mathcal{C} \) is a class of algebras such that for every pair \( \mathfrak{A}, \mathfrak{B} \in \mathcal{C} \), \( \mathfrak{A} \times \mathfrak{B} \) has directly decomposable tolerances, \( \mathcal{C} \) is said to have directly decomposable tolerances.

Let \( a \) and \( b \) be two elements of an algebra \( \mathfrak{A} \). \( T_A(a, b) \) denotes the least tolerance
on $\mathfrak{A}$ collapsing the pair $\langle a, b \rangle$, i.e. $T_A(a, b) = \bigcap\{T \mid T \in LT(\mathfrak{A})$ and $\langle a, b \rangle \in T\}$. Thus $T_A(a, b)$ is a generalization of the concept of a principal congruence.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras of the same type and $x$ an element of $\mathfrak{A} \times \mathfrak{B}$. When $x_1 = pr_1x$ and $x_2 = pr_2x$, $[x_1, x_2]$ is a componentwise denotation for $x$. Further, if $T_1 \in LT(\mathfrak{A})$ and $T_2 \in LT(\mathfrak{B})$, we have $\langle x, y \rangle \in T_1 \times T_2$ if $\langle x_1, y_1 \rangle \in T_1$ and $\langle x_2, y_2 \rangle \in T_2$. As noted in [2], the direct product of two tolerances is a tolerance on the direct product of the corresponding algebras.

1. TOLERANCES ON DIRECT PRODUCTS

The aim of this section is to give conditions under which the identity

$$\text{(1)} \quad (T_1 \times T_2) \lor_{A \times B} (S_1 \times S_2) = (T_1 \lor_A S_1) \times (T_2 \lor_B S_2)$$

is valid for two algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type and for every $T_1, S_1 \in LT(\mathfrak{A})$ and every $T_2, S_2 \in LT(\mathfrak{B})$. It is worth noting that (1) holds for congruences $T_1, T_2, S_1$ and $S_2$ on any algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type, see [3].

An algebra is called idempotent if for every $m$-ary polynomial $q(x_1, \ldots, x_m)$ over $\mathfrak{A}$ and for every $a \in A$, $q(a, \ldots, a) = a$. $\mathfrak{A}$ is called superidempotent if it is idempotent and for every $m$-ary polynomial $q$ and every two elements $a$ and $b$ of $\mathfrak{A}$ there are elements $c$ and $d$ such that $q(k, \ldots, k, a, k, \ldots, k) = a$ and $q(k, \ldots, k, b, k, \ldots, k) = b$, where $k$ is $c$ or $d$ according to the following rule: if $k$ on $i$th place is $c$ ($d$) in the expression for $a$ then $k$ on the $i$th place is also $c$ ($d$) in the expression for $b$, and vice versa. When $\mathfrak{A}$ is a lattice, it is superidempotent: the elements $c$ and $d$ corresponding to given $a$ and $b$ are $a \lor b$ and $a \land b$.

**Theorem 1.** Let $\mathcal{C}$ be a class of superidempotent algebras of the same type, where for every $\mathfrak{A} \in \mathcal{C}$ and every $T, S \in LT(\mathfrak{A})$, $\langle u, v \rangle \in T \lor_A S$ if and only if there exists a positive integer $N$ such that for every even $n > N$ there exist elements $u_i, v_i$ of $\mathfrak{A}$ ($i = 1, \ldots, n$) and an $n$-ary polynomial $p$ over $\mathfrak{A}$ with the properties

(i) $p(u_1, \ldots, u_n) = u$ and $p(v_1, \ldots, v_n) = v$;

(ii) $\langle u_i, v_i \rangle \in T$ for even and $\langle u_i, v_i \rangle \in S$ for odd values of $i$ ($i = 1, \ldots, n$).

Then the identity (1) is valid for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and every $T_1, S_1 \in LT(\mathfrak{A})$, $T_2, S_2 \in LT(\mathfrak{B})$.

**Proof.** The proof is a modification of the proof of [2, Thm. 1]. Evidently $T_1 \times T_2, S_1 \times S_2 \subseteq (T_1 \lor_A S_1) \times (T_2 \lor_B S_2)$, whence it remains to prove the inclusion $(T_1 \lor_A S_1) \times (T_2 \lor_B S_2) \subseteq (T_1 \times T_2) \lor_{A \times B} (S_1 \times S_2)$.

Let $a_1, b_1 \in A$ and $a_2, b_2 \in B$ be elements such that $\langle [a_1, a_2], [b_1, b_2] \rangle \in (T_1 \lor_A S_1) \times (T_2 \lor_B S_2)$. Then $\langle a_1, b_1 \rangle \in T_1 \lor_A S_1$ and $\langle a_2, b_2 \rangle \in T_2 \lor_B S_2$. According to the assumption, there are two positive integers $N_1$ and $N_2$ such that for every even integer $n > \max(N_1, N_2)$ there exist elements $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$. Thus $\langle u_1, v_1 \rangle, \ldots, \langle u_n, v_n \rangle \in (T_1 \lor_A S_1) \times (T_2 \lor_B S_2)$. Since $\mathfrak{A}$ and $\mathfrak{B}$ are superidempotent, there are $c \in A$ and $d \in B$ such that $\langle c, d \rangle \in T_1 \times T_2$. Therefore $\langle u_1, v_1 \rangle, \ldots, \langle u_n, v_n \rangle \in T_1 \times T_2$, and hence $\langle u_1, v_1 \rangle, \ldots, \langle u_n, v_n \rangle \in (T_1 \times T_2) \lor_{A \times B} (S_1 \times S_2)$.

Thus (1) holds for $\langle u_1, v_1 \rangle, \ldots, \langle u_n, v_n \rangle \in (T_1 \times T_2) \lor_{A \times B} (S_1 \times S_2)$.

Finally, let $a, b \in A$ and $c, d \in B$ be elements such that $\langle a, b \rangle \in T_1 \times T_2$. Then $\langle a, b \rangle \in T_1 \lor_A S_1$ and $\langle a, b \rangle \in T_2 \lor_B S_2$. According to the assumption, there are $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ such that $\langle [u_1, \ldots, u_n], [v_1, \ldots, v_n] \rangle \in (T_1 \lor_A S_1) \times (T_2 \lor_B S_2)$. Since $\mathfrak{A}$ and $\mathfrak{B}$ are superidempotent, there are $c \in A$ and $d \in B$ such that $\langle c, d \rangle \in T_1 \times T_2$. Therefore $\langle u_1, v_1 \rangle, \ldots, \langle u_n, v_n \rangle \in T_1 \times T_2$, and hence $\langle u_1, v_1 \rangle, \ldots, \langle u_n, v_n \rangle \in (T_1 \times T_2) \lor_{A \times B} (S_1 \times S_2)$.

Thus (1) holds for $\langle u_1, v_1 \rangle, \ldots, \langle u_n, v_n \rangle \in (T_1 \times T_2) \lor_{A \times B} (S_1 \times S_2)$.

Therefore (1) holds for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and every $T_1, S_1 \in LT(\mathfrak{A})$, $T_2, S_2 \in LT(\mathfrak{B})$. 

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of \( \mathfrak{A} \), \( u_1', \ldots, u_n', v_1', \ldots, v_n' \) of \( \mathfrak{B} \), and \( n \)-ary polynomials \( p \) and \( q \) over \( \mathfrak{A} \) and \( \mathfrak{B} \) having the properties

(i) \( p(u_1, \ldots, u_n) = a_1 \), \( p(v_1, \ldots, v_n) = b_1 \), \( q(u_1, \ldots, u_n) = a_2 \) and \( q(v_1, \ldots, v_n) = b_2 \);

(ii) \( \langle u_i, v_i \rangle \in T_1 \) and \( \langle u_i', v_i' \rangle \in T_2 \) for even and \( \langle u_i, v_i \rangle \in S_1 \) and \( \langle u_i', v_i' \rangle \in S_2 \) for odd values of \( i \), \( i = 1, \ldots, n \).

We define now an \( n^2 \)-ary polynomial \( r \) as follows: \( r(y_1, \ldots, y_{n^2}) = p(q(x_1, \ldots, x_n), \ldots, q(x_{n^2-1}, x_n)), \) where \( x_k, j = y_{(k-1)n+j} \). Let \( c_i \) and \( d_i \) be elements of \( \mathfrak{A} \) such that \( q(k_i, \ldots, k_i, u_i, k_i, \ldots, k_i) = u_i \) and \( q(k_i, \ldots, k_i, v_i, k_i, \ldots, k_i) = v_i \), where \( k_i \) is \( c_i \) or \( d_i \), \( i = 1, \ldots, n \). Furthermore, \( \langle c_i, c_i \rangle, \langle d_i, d_i \rangle \in S_1 \times T_1 \) for every value of \( i \). Accordingly, we can now write the following scheme:

- \( t \) and \( s \) are indices, \( t, s = 1, \ldots, n \);
- for each separate value of \( t \), \( z_s = w_s = k_t \) when \( s \neq t \), and \( z_s = u_s, w_s = v_s \) when \( s = t \);
- \( \langle [z_s, u_s], [w_s, v_s] \rangle \in T_1 \times T_2 \) when \( t \) is even and \( \langle [z_s, u_s], [w_s, v_s] \rangle \in S_1 \times S_2 \) when \( t \) is odd.

Moreover, \( r([z_1, u_1], [z_2, u_1], \ldots, [z_n, u_n]) = [r(z_1, \ldots, z_n, z_1, \ldots, z_n, \ldots, z_n)], \)

\( r(u_1', u_2', \ldots, u_n', u_2', \ldots, u_n') = [r(u_1, k_1, \ldots, k_1, k_1, u_2, k_2, \ldots, k_2, k_3, \ldots, u_n, k_n, u_2, \ldots, u_n)] = [a_1, a_2] \) and similarly \( r([w_1, v_1], [w_2, v_1], \ldots, [w_n, v_n]) = [b_1, b_2] \).

Obviously, \( \mathfrak{A} \times \mathfrak{B} \) is superidempotent when \( \mathfrak{A} \) and \( \mathfrak{B} \) are, whence one can derive from the polynomial \( r \) a new polynomial \( r^* \) such that \( r^* \) is \( (n^2 + j) \)-ary, where \( j \) is even and \( r^*(y_1, \ldots, y_{n^2+j}) = pr_1(r(y_1, \ldots, y_{n^2}), y_{n^2+1}, \ldots, y_{n^2+j}). \)

But then, according to the superidempotency of \( \mathfrak{A} \times \mathfrak{B} \) with respect to \( r^* \), \( [a_1, a_2] \) and \( [b_1, b_2] \), we have two sequences of elements and a polynomial \( r^* \) over \( \mathfrak{A} \times \mathfrak{B} \) such that (i) and (ii) hold for every even \( m > n^2 - 2 \) and thus \( \langle [a_1, a_2], [b_1, b_2] \rangle \in (T_1 \times T_2) \vee_{\mathfrak{A} \times \mathfrak{B}} (S_1 \times S_2) \). This completes the proof.

A join-semilattice \( \mathfrak{S} = (S, \vee) \) is called down directed, if for any two elements \( a, b \in S \) there is a common lower bound \( c \) of \( a \) and \( b \) in \( \mathfrak{S} \). An up directed meet-semilattice is defined dually. A quasilattice \( \mathfrak{Q} = (Q, \vee, \wedge) \) is a structure, where \( \vee \) and \( \wedge \) are commutative, associative and idempotent (see Plonka [4]), i.e. \( \mathfrak{Q} \) is a join-semilattice with respect to \( \vee \) and a meet-semilattice with respect to \( \wedge \). \( \mathfrak{Q} \) is a lattice if and only if the absorption laws hold in \( \mathfrak{Q} \). \( \mathfrak{Q} \) is down directed, if it is down directed as a join-semilattice, and up directed, if it is up directed as a meet-semilattice. Obviously, down directed join-semilattices and quasilattices as well as up directed meet-semilattices and quasilattices are superidempotent.

**Theorem 2.** Let \( \mathcal{C} \) be one of the following classes of algebras:

(i) the class of all lattices;

(ii) the class of all down directed join-semilattices;

(iii) the class of all up directed meet-semilattices;
the class of all down directed quasilattices;

Then (1) is true for each \( \mathcal{A} \), \( \mathcal{B} \in \mathcal{C} \) and for every \( T_1, S_1 \in LT(\mathcal{A}) \), \( T_2, S_2 \in LT(\mathcal{B}) \).

Proof. We have to show that when \( \mathcal{A} \in \mathcal{C} \), \( T, S \in LT(\mathcal{A}) \) and \( u, v \in A \), then \( \langle u, v \rangle \in T \lor_A S \) if and only if there exists an even positive integer \( N \) such that for every even integer \( n > N \) there are elements \( u_1, \ldots, u_n \) and \( v_1, \ldots, v_n \) and an \( n \)-ary polynomial \( p \) over \( \mathcal{A} \) such that (i) and (ii) of Theorem 1 hold. After proving this the assertion of the theorem follows from Theorem 1. We shall present the proof only for lattices; the proofs for (ii)–(v) are analogous and hence we omit them.

As proved in [1, Thm. 2], \( \langle u, v \rangle \in T \lor_A S \) if and only if there is a polynomial \( p^*(y_1, \ldots, y_m) \) and elements \( u_1^*, \ldots, u_m^* \) and \( v_1^*, \ldots, v_m^* \) such that \( \langle u_i^*, v_i^* \rangle \in T \) or \( \langle u_i^*, v_i^* \rangle \in S \), \( i = 1, \ldots, m \), \( p^*(u_1^*, \ldots, u_m^*) = u \) and \( p^*(v_1^*, \ldots, v_m^*) = v \). Thus if the conditions (i) and (ii) of Theorem 1 hold, then \( \langle u, v \rangle \in T \lor_A S \). So it remains to show the converse and we shall do it by modifying the polynomial \( p^*(y_1, \ldots, y_m) \) and the sequences \( u_1^*, \ldots, u_m^* \) and \( v_1^*, \ldots, v_m^* \) in a suitable manner.

Let us denote \( u_1^* \land \ldots \land u_m^* \land v_1^* \land \ldots \land v_m^* \) by \( a^* \). Trivially, \( \langle a^*, a^* \rangle \in T, S \), \( u_i^* \lor a^* = u_i^* \) and \( v_i^* \lor a^* = v_i^* \) for each \( i \), \( i = 1, \ldots, m \). If \( \langle u_i^*, v_i^* \rangle \in S \), we put \( u_1 = u_i^* \) and \( v_1 = v_i^* \), and if \( \langle u_i^*, v_i^* \rangle \notin S \), we put \( u_1 = v_i^* \), \( u_2 = u_i^* \) and \( v_2 = v_i^* \), clearly then \( \langle u_2, v_2 \rangle \in T \). Assume that \( \langle u_1^*, v_1^* \rangle \in S \), whence \( u_1 = u_1^* \) and \( v_1 = v_1^* \). If now \( \langle u_2^*, v_2^* \rangle \in T \), we put \( u_2 = u_2^* \) and \( v_2 = v_2^* \), and if \( \langle u_2^*, v_2^* \rangle \notin T \), then we put \( u_2 = v_2 = a^* \). In that case \( \langle u_2, v_2 \rangle \in T \) and because then \( \langle u_2^*, v_2^* \rangle \notin S \), we put \( u_2^* = u_3 \) and \( v_2^* = v_3 \). So from \( u_1^*, \ldots, u_m^* \), from \( v_1^*, \ldots, v_m^* \) and from \( a^* \) we can easily construct two new sequences \( u_1, \ldots, u_{2k} \) and \( v_1, \ldots, v_{2k} \) such that \( \langle u_i, v_i \rangle \in T \) for even and \( \langle u_i, v_i \rangle \in S \) for odd values of \( i \), \( i = 1, \ldots, 2k \). Assume that \( \langle u_i^*, v_i^* \rangle \notin S \), and so \( u_1 = v_1 = a^* \), \( u_2 = u_1^* \) and \( v_2 = v_1^* \). Then we replace \( y_1 \) in the polynomial \( p^*(y_1, \ldots, y_m) \) by the expression \( x_1 \lor x_2 \) and obtain a new polynomial \( p'(x_1, x_2, y_2, \ldots, y_m) \). After performing all similar necessary modifications in the polynomial \( p^* \) we have a new one: \( p(x_1, \ldots, x_{2k}) \). Because \( u_i^* \lor a^* = u_i^* \) and \( v_i^* \lor a^* = v_i^* \), \( p(u_1^*, \ldots, u_{2k}) = u \) and \( p(v_1^*, \ldots, v_{2k}) = v \). Now we may put \( N = 2k + 2 \) and if \( n > N \), we put \( u_i = v_i = a^* \) for \( i = 2k + 1, \ldots, n \), and moreover \( p(x_1, \ldots, x_n) = p(x_1, \ldots, x_{2k}) \lor x_{2k+1} \lor \ldots \lor x_n \). In this case the conditions (i) and (ii) of Theorem 1 also hold, and the required result follows from [1, Thm. 2].

It is proved in [2] that the identity (1) implies a similar identity for an arbitrary number of tolerances on direct factors, i.e.

\[
(2) \quad \bigvee_{A \times B \times \mathcal{T}} T \times S \mid \gamma \in \Gamma = \bigvee_A \{T \mid \gamma \in \Gamma\} \times \bigvee_B \{S \mid \gamma \in \Gamma\}
\]

(\( \Gamma \) is an arbitrary index set) in the class of all distributive lattices with a least and a greatest element as well as in the class of all monoids with a unit element. In the following we extend this result. The proof follows from that of [2, Thm. 2], where the unit element is substituted by a lower bound (by an upper bound) of the elements under consideration and the operation \( \circ \) by \( \lor \) (by \( \land \)). Hence the proof is omitted.
Theorem 3. Let $\mathcal{C}$ be one of the classes (i)–(v) of algebras in Theorem 2. Then (2) is valid for each pair $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and for every $T_i \in \text{LT}(\mathfrak{A})$, $S_i \in \text{LT}(\mathfrak{B})$, where $I$ is an arbitrary index set.

2. DIRECT DECOMPOSABILITY

Theorem 5 of [2] can be generalized in the following way:

**Theorem 4.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras of the same type satisfying (2). Then the following conditions are equivalent:

1. $\mathfrak{A} \times \mathfrak{B}$ has directly decomposable tolerances;
2. $\langle a, b \rangle \in T$ implies $T_\mathfrak{A}(a_1, b_1) \times T_\mathfrak{B}(a_2, b_2) \subseteq T$ for each $T \in \text{LT}(\mathfrak{A})$.

**Proof.** (1) $\Rightarrow$ (2). The equality $T_\mathfrak{A} \times \mathfrak{B} (a, b) = T_1 \times T_2$ evidently implies that $T_\mathfrak{A}(a_1, b_1) \subseteq T_1$, $T_\mathfrak{B}(a_2, b_2) \subseteq T_2$, and thus $\langle a, b \rangle \in T$ implies that $T_\mathfrak{A}(a_1, b_1) \times T_\mathfrak{B}(a_2, b_2) \subseteq T_1 \times T_2 \subseteq T$.

(2) $\Rightarrow$ (1). Let $T_1 = \{\langle a_1, b_1 \rangle \mid \text{there exist } a_2, b_2 \text{ of } \mathfrak{B} \text{ such that } \langle [a_1, a_2], [b_1, b_2] \rangle \in T \}$ and $T_2 = \{\langle a_2, b_2 \rangle \mid \text{there exist } a_1, b_1 \text{ of } \mathfrak{A} \text{ such that } \langle [a_1, a_2], [b_1, b_2] \rangle \in T \}$.

By Theorem 14 in [1], $T_1 = \bigvee \{T_\mathfrak{A}(a_1, b_1) \mid \langle a, b \rangle \in T \}$ and $T_2 = \bigvee \{T_\mathfrak{B}(a_2, b_2) \mid \langle a, b \rangle \in T \}$. Then it follows from (2) that $T_1 \times T_2 = (\bigvee \{T_\mathfrak{A}(a_1, b_1) \mid \langle a, b \rangle \in T \}) \times (\bigvee \{T_\mathfrak{B}(a_2, b_2) \mid \langle a, b \rangle \in T \}) \subseteq \bigvee \{T_\mathfrak{A}(a_1, b_1) \times T_\mathfrak{B}(a_2, b_2) \mid \langle a, b \rangle \in T \} \subseteq T$. The converse inclusion is evident. Because $T_1 \in \text{LT}(\mathfrak{A})$ and $T_2 \in \text{LT}(\mathfrak{B})$, (1) is proved.

Next we shall prove two lemmas, by means of which we can prove the second result from the introduction.

**Lemma 1.** Let $\mathfrak{A} = (A, F)$ be an algebra and $a, b \in A$. $\langle x, y \rangle \in T_\mathfrak{A}(a, b)$ if and only if there exists a binary algebraic function $\varphi$ over $\mathfrak{A}$ such that $x = \varphi(a, b)$ and $y = \varphi(b, a)$.

**Proof.** Clearly the set of all pairs $\langle x, y \rangle$ for all binary algebraic functions $\varphi$ from the theorem over $\mathfrak{A}$ constitute a reflexive and symmetric binary relation $T$ having the Substitution Property and collapsing $\langle a, b \rangle$, i.e. $T_\mathfrak{A}(a, b) \subseteq T$. The converse inclusion is evident.

**Lemma 2.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be two lattices. Then $T_\mathfrak{A}(a_1, b_1) \times T_\mathfrak{B}(a_2, b_2) \subseteq T_\mathfrak{A} \times \mathfrak{B}(a, b)$ for every pair $\langle a, b \rangle$ of elements of $\mathfrak{A} \times \mathfrak{B}$.

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Proof. Let \( \langle x, y \rangle \in T_A(a_1, b_1) \times T_B(a_2, b_2) \). Then \( \langle x_1, y_1 \rangle \in T_A(a_1, b_1) \), \( \langle x_2, y_2 \rangle \in T_B(a_2, b_2) \) and, according to Lemma 1, there exist \((2 + n)\)-ary and \((2 + m)\)-ary polynomials \( p \) and \( q \) such that \( x_1 = p(a_1, b_1, c_1, \ldots, c_n) \), \( y_1 = p(b_1, a_1, c_1, \ldots, c_n) \), \( x_2 = q(a_2, b_2, d_1, \ldots, d_m) \) and \( y_2 = q(b_2, a_2, d_1, d_2, \ldots, d_m) \). Let \( s = \max(m, n) \) and let us put \( c_i = c_n \) and \( d_j = d_m \) for \( i = n, \ldots, s \) and \( j = m, \ldots, s \). Now we can construct a \((4 + s)\)-ary polynomial \( r \) as follows: \( r(x, y, k_1, \ldots, k_s) = e_1 \land p(x, y, k_1, \ldots, k_s) \) and \( r(x, y, l_1, \ldots, l_s) = e_2 \land q(x, y, k_1, \ldots, k_s) \). But then \( p(x, y, c_1, \ldots, c_n) = r(x, y, c_1, \ldots, c_n, h, g) \) and \( q(x, y, d_1, \ldots, d_m) = r(x, y, d_1, \ldots, d_m, g, h) \), where \( h = x \lor y \lor c_1 \lor \ldots \lor c_n \lor d_1 \lor \ldots \lor d_m \) and \( g = x \land y \land c_1 \land \ldots \land c_n \land d_1 \land \ldots \land d_m \). Further, \( \langle x, y \rangle = \langle [x_1, x_2], [y_1, y_2] \rangle = \langle [r(a_1, b_1, c_1, \ldots, c_n, h, g)], [r(b_1, a_1, c_1, \ldots, c_n, h, g)], [r(b_2, a_2, d_1, \ldots, d_m, g, h)], [r(b_2, a_2, d_1, \ldots, d_m, g, h)] \rangle = \langle r(a, b, [c_1, d_1], \ldots, [c_n, d_m], [h, g], [g, h]), r(a, b, [c_1, d_1], \ldots, [c_n, d_m], [h, g], [g, h]) \rangle \), where \( \phi(x, y) \) = \( r(x, y, [c_1, d_1], \ldots, [c_n, d_m], [h, g], [g, h]) \). According to Lemma 1, \( \langle x, y \rangle \in T_A \times T_B(a, b) \). This completes the proof.

Now we can prove

**Theorem 5.** *The class of all lattices has directly decomposable tolerances.*

**Proof.** By Theorem 3, the class from the theorem satisfies the identity (2), and thus Theorem 4 can be used. According to Lemma 2, 2) of Theorem 4 holds, whence the proof is a direct consequence of Theorem 4.

**References**


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