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REMARKS ON THE INTERPOLATION OF ANISOTROPIC SPACES OF BESOV-HARDY-SOBOLEV TYPE

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This paper deals with the interpolation of anisotropic spaces $B_{p,q}^s$ and $F_{p,q}^s$ of Besov-Hardy-Sobolev type, which are introduced in [11], [9] (cf. also [12]). These spaces generalize the classical anisotropic Besov spaces, Sobolev spaces and Hardy spaces (cf. [2]), as we shall show in Section 3. The fundamental ideas are the same as those used when approaching the interpolation of the corresponding isotropic spaces of Besov-Hardy-Sobolev type, which are due to J. Peetre (cf. [5]) and H. Triebel (cf. [11], [13]).

1. DEFINITIONS AND BASIC PROPERTIES OF THE SPACES

The definition of the anisotropic spaces of Besov-Hardy-Sobolev type is based on an anisotropic decomposition in the Fourier image of the distributions considered with the aid of decomposition-functions. For these functions we need an anisotropic decomposition of $\mathbb{R}^n$ (Euclidean n-space).

Let $a := (a_1, \ldots, a_n)$ be a fixed n-tuple of positive numbers. Then we subdivide the corridors

$$K_k := \{x \mid |x_j| \leq 2^{ka_j}, j = 1, \ldots, n\} \setminus \{x \mid |x_j| < 2^{(k-1)a_j}, j = 1, \ldots, n\}, \quad k = 1, 2, 3, \ldots, \quad x := (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

in a natural way by the hyperplanes $\{x \mid x_j = 0\}$, $\{x \mid x_j = 2^{(k-1)a_j}\}$ and $\{x \mid x_j = -2^{(k-1)a_j}\}, j = 1, \ldots, n$, into closed rectangles $P_{k,t}$, $t = 1, \ldots, T (T = 4^n - 2^n)$.

Furthermore, we set

$$K_0 := \{x \mid |x_j| \leq 1, j = 1, \ldots, n\}$$

and for simplicity, $P_{0,t} := K_0$ for $t = 1, \ldots, T$.

In addition to the $P_{k,t}$ we consider a little larger rectangle $P_{k,t}^*$, with the same centre as $P_{k,t}$ and sides parallel to the corresponding sides of $P_{k,t}$, which are all $\kappa$-times larger, $1 < \kappa < 1 + 2^{1-\max a_i}$, so that

$$P_{k,t}^* \subset (K_{k-1} \cup K_k \cup K_{k+1}), \quad t = 1, \ldots, T$$

(modification for $k = 0$).
The decomposition
\[ \mathcal{P} := \{ P_{k,t} \}, \quad \mathbb{R}^n = \bigcup_{k=0,1,2,\ldots, t=1,\ldots,T} P_{k,t}, \]
is a regular covering of \( \mathbb{R}^n \) in the sense of [9].

Remark 1. If we take in (1) corridors \( K_k \) for \( k = 0, \pm 1, \pm 2, \ldots \) then we get by the same construction the so-called anisotropic homogeneous decomposition
\[ \mathcal{P}^\sim := \{ P_{k,t} \}, \quad \mathbb{R}^n - \{ 0 \} = \bigcup_{k=0,\pm 1,\pm 2,\ldots, t=1,\ldots,T} P_{k,t}. \]

Remark 2. In addition to the decomposition \( \mathcal{P} \) of \( \mathbb{R}^n \) we shall later consider the so-called "local modification" \( \mathcal{P}^\sim \) of this decomposition. This means: \( \mathcal{P}^\sim := \{ \bar{P}_{k,t} \}, \) \( \bar{P}_{k,t} \) are rectangles with sides parallel to the co-ordinate axes, \( k = 0, 1, 2, \ldots, t = 1, \ldots, T \) (\( T \) a fixed natural number),
\[ \mathbb{R}^n = \bigcup_{k=0,1,1,\ldots, t=1,\ldots,T} \bar{P}_{k,t}, \quad \bar{P}_{0,t} := K_0 \]
and there is a fixed natural number \( N \) such that
\[ \bar{P}_{k,t} \subset \bigcup_{h=-N}^{N} K_{k+h} \]
for an arbitrary \( k \) and \( t = 1, \ldots, T \) (\( K_l := 0 \) for \( l < 0 \)).

Now we come to the system of decomposition-functions. \( S = S(\mathbb{R}^n) \) denotes the Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions on \( \mathbb{R}^n \), and \( S' = S'(\mathbb{R}^n) \) the corresponding dual space of tempered distributions. \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier transform and its inverse transform on \( S'(\mathbb{R}^n) \).

If \( L \) is a natural number then \( \mathcal{A}_L = \mathcal{A}_L[S] \) is the set of all systems \( \varphi := \{ \varphi_{k,t} \} \), such that
\[ \varphi_{k,t} \in S, \quad \text{supp} \varphi_{k,t} \subset P_{k,t}, \quad k = 0, 1, 2, \ldots, t = 1, \ldots, T, \]
and
\[ \sup_{k=0,1,2,\ldots} \sup_{x \in \mathbb{R}^n} \left[ \sum_{j=1}^{n} (1 + x_j^2)^{1/2} |a \cdot \alpha| \right] |D^\alpha \varphi_{k,t}(x)| = c_\varphi < \infty, \]
where
\[ a \cdot \alpha := \sum_{j=1}^{n} a_j \alpha_j, \quad \alpha := (\alpha_1, \ldots, \alpha_n), \quad \alpha_j \geq 0 \quad \text{integer}, \]
\[ D^\alpha \varphi_{k,t} := \frac{\partial^{|\alpha|} \varphi_{k,t}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \]

Furthermore, let
\[ \sum_{k=0,1,2,\ldots, t=1,\ldots,T} \varphi_{k,t}(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n. \]
It is easy to see that \( \mathcal{A}_L^0 \) is not empty.
After these preliminaries we are able to define the anisotropic spaces $B_{p,q}^s$ and $F_{p,q}^s$.

We use the following abbreviations: For measurable functions $f$ on $\mathbb{R}^n$,

$$
\|f\|_{L_p} := \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p} \quad \text{if } 0 < p < \infty ,
$$

$$
\|f\|_{L_\infty} := \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|
$$

and for sequences $\{g_{k,t}(x)\}_{k=0,1,2,\ldots}$ of measurable functions,

$$
\|g_{k,t}\|_{L_q(L_p)} := \left( \sum_{k=0}^{\infty} \sum_{t=1}^{T} \|g_{k,t}\|_{L_p}^q \right)^{1/q} \quad \text{if } 0 < q < \infty , \quad 0 < p \leq \infty ,
$$

$$
\|g_{k,t}\|_{L_\infty(L_p)} := \sup_{k=0,1,2,\ldots} \|g_{k,t}\|_{L_p} , \quad 0 < p \leq \infty ,
$$

$$
\|g_{k,t}\|_{L_p(L_q)} := \left( \sum_{k=0}^{\infty} \sum_{t=1}^{T} |g_{k,t}(x)|^q \right)^{1/q} \quad \text{if } 0 < q < \infty , \quad 0 < p \leq \infty ,
$$

$$
\|g_{k,t}\|_{L_\infty(L_q)} := \sup_{k=0,1,2,\ldots} |g_{k,t}(x)| \quad \text{if } 0 < q < \infty , \quad 0 < p \leq \infty .
$$

**Definition.** Let $a := (a_1, \ldots, a_n)$ be the $n$-tuple of positive numbers which characterizes the decomposition $\mathcal{A}$ and $s := (s/a_1, \ldots, s/a_n)$, $-\infty < s < \infty$.

(i) If $0 < p \leq \infty$, $0 < q \leq \infty$ and if $L$ is a fixed natural number, $L > L^\mathcal{A}(s, p, q)$, then

$$
B_{p,q}^s := \{f \in S' : \|f\|_{B_{p,q}^s}^\varphi := \|2^{sk}\mathcal{F}^{-1}\varphi_{k,t} f\|_{L_q(L_p)} < \infty \text{ for all } \varphi \in \mathcal{A}_L^0 \}. 
$$

(ii) If $0 < p < \infty$, $0 < q \leq \infty$ and if $L$ is a fixed natural number, $L > L^\mathcal{A}(s, p, q)$, then

$$
F_{p,q}^s := \{f \in S' : \|f\|_{F_{p,q}^s}^\varphi := \|2^{sk}\mathcal{F}^{-1}\varphi_{k,t} f\|_{L_p(L_q)} < \infty \text{ for all } \varphi \in \mathcal{A}_L^0 \}. 
$$

(iii) If $0 < p < \infty$, then

$$
H_p^s := F_{p,2}^s .
$$

**Remark 3.** In our definition we have used the sets $S(\mathbb{R}^n)\cdot S'(\mathbb{R})$ instead of $Z(\mathbb{R}^n)$ and $Z'(\mathbb{R}^n)$, respectively, which were used in the paper [9]. Both definitions coincide, cf. [13], p. 27. It is easy to see that the definition of $B_{p,q}^s$ and $F_{p,q}^s$ depends only on the quotients $s/a_1, \ldots, s/a_n$; this justifies our notation.

We recall the basic properties of $B_{p,q}^s$ and $F_{p,q}^s$ which are proved in [9]. $B_{p,q}^s$ equipped with the quasi-norm $\|f\|_{B_{p,q}^s}$ is a quasi-Banach space (Banach space if $1 \leq p \leq \infty$).

1) The numbers $L^f$ and $L^B$ can be chosen (cf. [12], p. 82) so that

$$
L^B(s, p, q) := |s| + \frac{6n}{p} + n + 4 , \quad L^f(s, p, q) := |s| + \frac{6n}{\min(p, q)} + n + 4 .
$$

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and \(1 \leq q \leq \infty\). All the quasi-norms \(\|f\|_{p,q}^\varphi\) with \(\varphi \in \bigcup_{L>1} L_B^m\) are mutually equivalent; so we omit the symbol \(\varphi\) in \(\|\cdot\|_{p,q}^\varphi\).

The corresponding assertions hold for the spaces \(F_{p,q}^s\) \((p < \infty)\). For all admissible values of parameters we have

\[
S \subset B_{p,q}^s \subset S', \quad S \subset F_{p,q}^s \subset S',
\]

where the sign \(\subset\) means a topological embedding.

Remark 4. If we replace the nonhomogeneous decomposition \(\mathcal{D}\) of \(\mathbb{R}^n\) by the homogeneous decomposition \(\mathcal{D}\) of \(\mathbb{R}^n - \{0\}\), take the system of decomposition-functions from \(D(\mathbb{R}^n - \{0\})\) instead from \(S(\mathbb{R}^n)\) and replace the condition (5) by

\[
\sup_{k=\pm 1, \pm 1, \ldots} \sup_{x \in \mathbb{R}^n} \left| \sum_{j=1}^n x_j^{1/\alpha_j} \left| D^2 \varphi_{k,t}(x) \right| \right| = c_\varphi < \infty,
\]

then we get in the same way as above the homogeneous anisotropic spaces

\[
\dot{B}_{p,q}^s, \dot{F}_{p,q}^s, \dot{H}_{p}^s := \dot{F}_{p,2}^s
\]

consisting of distributions from \(Z'(\mathbb{R}^n - \{0\})\) instead of \(S'(\mathbb{R}^n)\). Here \(Z'(\mathbb{R}^n - \{0\})\) is the strong topological dual of the Fourier image \(Z(\mathbb{R}^n - \{0\})\) of the space \(D(\mathbb{R}^n - \{0\})\) (complex-valued infinitely differentiable functions with compact supports in \(\mathbb{R}^n - \{0\}\) equipped in the usual way with a locally convex topology). For these spaces we also have the corresponding basic properties as above, cf. [9].

Remark 5. Let \(\dot{B}_{p,q}^s, \dot{F}_{p,q}^s, \dot{H}_{p}^s\) be the spaces which are analogously defined as \(B_{p,q}^s, F_{p,q}^s, H_{p}^s\), based on a local modified decomposition \(\mathcal{D}\) of \(\mathcal{D}\) (cf. Definition 4 in [9]). Then it is easy to see that also in the anisotropic case the identities

\[
\dot{B}_{p,q}^s = B_{p,q}^s, \quad \dot{F}_{p,q}^s = F_{p,q}^s
\]

hold for all admissible values of \(s, p, q\) (for the isotropic case cf. [13], p. 43).

Finally, we remark that the definitions for \(a_1 = a_2 = \ldots = a_n > 0\) yield the corresponding isotropic spaces. If the above relation is not fulfilled, then the result are anisotropic spaces even for \(s = 0\). It is nontrivial to get isotropic spaces in the same cases, for instance: \(H_{p}^0 = L_p, 1 < p < \infty\).

2. INTERPOLATION THEOREMS

The symbol \((\cdot, \cdot)_{p,q}\) denotes the K-method of interpolation (cf. [11], Sec. 1.3.2 or [1], Sec. 3.1, and for the extension to the quasi-Banach spaces [1], Sec. 3.11).

For general (anisotropic) \(H^p\) spaces (including the usual Hardy spaces), A. P. 236
Calderon and A. Torchinsky proved in [2, II] the following interpolation theorem:

\[(9) \quad (H^{p_0}, H^{p_1})_{\theta, p} = H^p,\]

where

\[\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < p_0 < p_1 < \infty, \quad 0 < \theta < 1.\]

H. Triebel showed in [14] that in a special case these general \(H^p\) spaces coincide with the anisotropic spaces \(\hat{H}^p_{p_0}\) from our definition (cf. Remark 4). So (9) yields the result for \(0 < p_0 < p_1 < \infty, 0 < \theta < 1,\)

\[(10) \quad (H^{p_0}_{p_0}, \hat{H}^p_{p_1})_{\theta, p} = \hat{H}^p_{p}, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},\]

where all the spaces have the same underlying decomposition of \(\mathbb{R}^n - \{0\}\).

This result for homogeneous spaces implies the corresponding result for non-homogeneous spaces.

**Proposition 1.** If \(0 < p_0 < p_1 < \infty\) and \(0 < \theta < 1,\) then

\[(11) \quad (H^{p_0}_{p_0}, H^{p_1}_{p_1})_{\theta, p} = H^p, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},\]

where all the spaces have the same underlying decomposition of \(\mathbb{R}^n\).

**Proof.** Let \(\psi\) be a function belonging to \(S\) with the properties:

\[\text{supp} \psi \subset K_2 := \{x \mid |x| \leq 2\} \quad \text{and} \quad \psi(x) = 1 \quad \text{for all} \quad x \in \{x \mid |x_j| \leq 1, j = 1, \ldots, n\}.\]

Then we split the functions \(f \in H^p_p\) into

\[f = \mathcal{F}^{-1} \psi \mathcal{F} f + \mathcal{F}^{-1} (1 - \psi) \mathcal{F} f = f^{(0)} + f^{(1)},\]

where \(f^{(0)} \in H^0_{p_0}\) and \(f^{(1)} \in H^0_{p_1} \cap \hat{H}^p_{p}.\)

For the \(\mathcal{K}\)-functional of the interpolation method

\[\mathcal{K}(t, f, H^0_{p_0}, H^0_{p_1}) := \inf_{f = f_0 + f_1} \{ \|f_0\|_{H^0_{p_0}} + t \|f_1\|_{H^0_{p_1}} \}, \quad 0 < t < \infty,\]

we have by this splitting of \(f = f_0 + f_1 \in (H^0_{p_0} + H^0_{p_1}),\)

\[f = f_0^{(0)} + f_1^{(0)} + f_0^{(1)} + f_1^{(1)} = (f_0^{(0)} + f_1^{(0)}) + (f_0^{(1)} + f_1^{(1)}) = f^{(0)} + f^{(1)},\]

\[(12) \quad \mathcal{K}(t, f, H^0_{p_0}, H^0_{p_1}) \sim \mathcal{K}(t, f^{(0)}, H^0_{p_0}, H^0_{p_1}) + \mathcal{K}(t, f^{(1)}, H^0_{p_0}, H^0_{p_1}).\]

Consider the term on the right-hand side. It is clear that

\[(13) \quad \mathcal{K}(t, f^{(1)}, H^0_{p_0}, H^0_{p_1}) = \mathcal{K}(t, f^{(1)}, \hat{H}^0_{p_0}, \hat{H}^0_{p_1})\]

and

\[(14) \quad \mathcal{K}(t, f^{(0)}, H^0_{p_0}, H^0_{p_1}) \sim \mathcal{K}(t, f^{(0)}, L^2_{p_{K_2}}, L^2_{p_{K_2}}),\]

where \(L^p_p := \{g \mid g \in S', \text{ supp } \mathcal{F} g \subset \Omega, \|g\|_{L^p_p} := \|\mathcal{F} g\|_{L_p} < \infty\}.\)
Further, for an appropriate $h \in \mathbb{R}^n$ we have
\[ f^{(0)}(x) := f^{(0)}(x) e^{i\theta h} \in (L^2_{p_0}, L^2_{p_1}) \] with $\Omega \cap K_2 = \emptyset$, and
\[ \|f^{(0)}\|_{L^p_{p_i}} \sim \|f^{(0)}\|_{L^p_{p_i}} \sim \|f^{(0)}\|_{H^0_{p_i}} \sim \|f^{(0)}\|_{H^0_{p_i}}, \quad i = 0, 1. \]
So we have
\[ \mathcal{K}(t, f^{(0)}, H^0_{p_i}, H^0_{p_i}) \sim \mathcal{K}(t, f^{(0)}, L^0_{p_0}, L^0_{p_1}) \sim \mathcal{K}(t, f^{(0)}, H^0_{p_0}, H^0_{p_1}). \]
From (12), (13), (14) and (10), (15) we finally get
\[ \|f\|_{(H^0_{p_0}, H^0_{p_1})_{\theta,p}} \sim \|f^{(0)}\|_{(H^0_{p_0}, H^0_{p_1})_{\theta,p}} + \|f^{(1)}\|_{(H^0_{p_0}, H^0_{p_1})_{\theta,p}} \]
\[ \sim \|f^{(0)}\|_{H^0_{p_0}} + \|f^{(1)}\|_{H^0_{p_0}} \sim \|f^{(0)}\|_{L^p_{p_0}} + \|f^{(1)}\|_{H^0_{p_0}} \]
\[ \sim \|f\|_{H^0_{p_0}}, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}. \]
The construction of a retraction from the space $B^s_{p,q}$ into the vector-valued sequence space $l^q(A)$, the known interpolation theorems for $l^q(A)$ and the proposition above yield the following interpolation theorem for the spaces $B^s_{p,q}$ and $F^s_{p,q}$.

**Theorem 1.** Let $a = (a_1, \ldots, a_n)$ be a fixed $n$-tuple of positive numbers which characterizes the anisotropic decomposition for all considered spaces, $s_i := s_i(a_1, \ldots, a_n), \infty < s_i < \infty (i = 0, 1)$ and $0 < \theta < 1$.

(i) If $0 < p \leq \infty$, $0 < q_0, q_1, q \leq \infty$, $s_0 = s_1$, then
\[ (B^s_{p_0, q_0}, B^s_{p_1, q_1})_{\theta,q} = B^s_{p,q}, \]
and for $p < \infty$,
\[ (F^s_{p_0, q_0}, F^s_{p_1, q_1})_{\theta,q} = (F^s_{p_0, q_0}, B^s_{p_1, q_1})_{\theta,q} = B^s_{p,q}, \]
where $s := (1 - \theta) s_0 + \theta s_1$.

(ii) If $0 < p_0 < p_1 < \infty$, $0 < q_0 < q_1 < \infty$, then
\[ (B^s_{p_0, q_0}, B^s_{p_1, q_1})_{\theta,p} = B^s_{p,p}, \]
where
\[ s := (1 - \theta) s_0 + \theta s_1 \quad \text{and} \quad \frac{1}{p} := \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}. \]

**Proof.** Step 1. First we recall a useful fact, given by H. Triebel in [15].

\[ l^q(A) := \{ \xi \mid \xi = (\xi_k, t)_{k=0,1,2, \ldots}, \| \xi \|_{l^q(A)} := (\sum_{k=0}^{\infty} \sum_{t=1}^{T} 2^{ktq} \| \xi_k, t \|_A)^{1/q} < \infty \}. \]
For $0 < p < \infty$, $\varphi = \{\varphi_{k,t}\} \in A_0^0 (L$ large enough) and all $f \in S'$, the inequality
\[
\begin{aligned}
c_i \|\mathcal{F}^{-1} \varphi_{k,t} \mathcal{F} f\|_{L_p} &\leq \|\mathcal{F}^{-1} \varphi_{k,t} \mathcal{F} f\|_{L_p^0}^0 \leq c_2 \|\mathcal{F}^{-1} \varphi_{k,t} \mathcal{F} f\|_{L_p} \\
k = 0, 1, 2, \ldots, \quad t = 1, \ldots, T,
\end{aligned}
\] holds with constants $c_1, c_2 > 0$ independent of $k, t$.

Hence in the definition of $B^p_{r,q}$ we can replace the space $L^q_p(L_\omega)$ by the space $L^q_p(H^0_p)$, $0 < p < \infty$, which is very useful, because for $H^0_p$ we have the Fourier transform and multiplier theorems.

**Step 2.** Now we establish that for $0 < p_0, p_1 < \infty$, $0 < q_0 \leq q_1 \leq \infty$, the following relation holds:
\[
\|f\|_{(B^{\mathcal{P}_0}_{p_0} \cap B^{\mathcal{P}_1}_{p_1,q_1})_{e,q}} \sim \|\mathcal{F}^{-1} \varphi_{k,t} \mathcal{F} f\|_{(l_q^0(A_0) \cap l_q^1(A_1))_{e,q}},
\]
where $A_i := H^0_{p_i}$ if $p_i < \infty$ and $A_i := L_\omega$ if $p_i = \infty$ ($i = 0, 1$).

The equivalence of the quasi-norms in (21) follows from Theorem 1.2.4 in [11] (which also holds for quasi-Banach spaces), if we have a retraction $R$ from $B^{q_1}_{p,q}$ into $l_q^0(H^0_p)$ or $l_q^1(L_\omega)$, respectively. Now we construct such an operator $R$.

Let $\varphi = \{\varphi_{k,t}\}$ be a system of decomposition functions corresponding to the decomposition $\mathcal{P}$, $\varphi \in A_0^0(\mathcal{P})$. Then there is a local modification $\hat{\mathcal{P}}$ of $\mathcal{P}$ and a corresponding system $\psi = \{\psi_{k,t}\} \in A_0^0(\mathcal{P})$, cf. [9], Definition 2, such that
\[
\psi_{k,t}(x) = 1 \quad \text{for} \quad x \in \text{supp} \, \varphi_{k,t}
\]
and
\[
\psi_{k,t}(x) = \psi(\{2^{-k\alpha}(x_j - \tilde{x}^{k,t}_j)\}) \quad \text{for all} \quad x \in \mathbb{R}^n
\]
($\tilde{x}^{k,t}$ is the centre of $\tilde{P}_{k,t} \in \mathcal{P}$, $\psi$ an appropriate function from $S$). With the aid of these systems we define the linear operators
\[
\mathcal{S} f := \{\mathcal{F}^{-1} \varphi_{k,t} \mathcal{F} f\}_{k=0,1,2,\ldots}, \quad f \in B^p_{r,q},
\]
and
\[
\mathcal{R} \{g_{1,n}\} := \sum_{n=0,1,2,\ldots} \mathcal{F}^{-1} \psi_{1,n} \mathcal{F} g_{1,n}, \quad \{g_{1,n}\} \in l_q^0(H^0_p).
\]

i) $\mathcal{S}$ is a bounded linear operator from $B^p_{r,q}$ into $l_q^0(H^0_p)$. This follows immediately from (20).

ii) $\mathcal{R}$ is a bounded linear operator from $l_q^0(H^0_p)$ into $B^p_{r,q}$. We get this from the estimate
\[
\|\mathcal{R} \{g_{1,n}\}\|_{B^p_{r,q}} \leq \|\{2^{sk} \mathcal{F}^{-1} \psi_{k,t} \mathcal{F} (\mathcal{R} \{g_{1,n}\})\}_{k=0,1,2,\ldots}\|_{l_q^0(H^0_p)} \leq \|\{2^{sk} \mathcal{F}^{-1} \sum_{l=0,1,2,\ldots} \mathcal{F} \psi_{1,n} \mathcal{F} g_{1,n}\}_{k,}\|_{l_q^0(H^0_p)} \leq \|\sum_{k=0,1,2,\ldots} 2^{sk} g_{k,1} \|_{l_q^0(H^0_p)} \leq \|\{2^{sk} \mathcal{F}^{-1} \psi_{k,t} \mathcal{F} (\mathcal{R} \{g_{1,n}\})\}_{k,}\|_{l_q^0(H^0_p)} \leq \|\sum_{k=0,1,2,\ldots} 2^{sk} g_{k,1} \|_{l_q^0(H^0_p)} \leq c\|g_{k,1}\|_{l_q^0(H^0_p)}.
\]
where (25) is established by the multiplier property of $\psi$, (cf. [9], Theorem 5) for the spaces $F_{p,2} = F_{p,2}^0$ (cf. Remark 5) with a constant which is independent of $k, t$.

iii) $R F f = \sum_{t=0,1,2,\ldots} F^{-1} \psi_{l,n} \varphi_{l,n} F f = \sum_{t=0,1,2,\ldots} F^{-1} \psi_{l,n} F f = f$ for all $f \in B_{p,q}^s$.

This means $R$ and $F$ are a retraction and a coretraction for the spaces $B_{p,q}^s$ and $l_q^0(H_p^0)$, where $0 < p < \infty$, $0 < q \leq \infty$, $-\infty < s < \infty$. In the case $p = \infty$ the operators $R$ and $F$ are a retraction and a coretraction for the spaces $B_{\infty,q}^s$ and $l_q^0(L_{\infty})$. Instead of estimating by a multiplier theorem in (25), we use now the estimate

$$
|F^{-1}(\psi_{k,t} \varphi_{k+h,t+i} F g_{k+h,t+i}^l(x))|
= |\left[ F^{-1}(\psi(2^{-k+h}) \varphi(2^{-k+h} \alpha_j)) \ast g_{k+h,t+i}^l \right](x)|
= 2^{k+h} \int_{R^n} F^{-1}(\psi(\varphi(2^{-k+h} \alpha_j))) \{2^{k+h}(x_j - \tilde{x}_j^l, 1)\} g_{k+h,t+i}^l(y) dy \leq c \|g_{k+h,t+i}^l\|_{L_{\infty}}.
$$

Step 3. The concrete interpolation formula (17) follows now from (21) and the known interpolation theorem (cf. [1], p. 122, extended to quasi-Banach spaces $A$)

$$(l_{q_0}^0(A), l_{q_1}^0(A))_{s, q} = l_{q}^0(A), \quad s = (1 - \theta) s_0 + \theta s_1,$$

with $s = (s_0, s_1) \in (0, \infty)$, $0 < \theta < 1$ and $A := H_p^0$ for $0 < p < \infty$ and $A := L_{\infty}$ for $p = \infty$. Formula (18) is obtained from (17) and (8) by the reiteration theorem (cf. [1], p. 50, extended to quasi-Banach spaces). Formula (19) follows from (21) and the interpolation theorem (cf. [1], p. 123, extended to quasi-Banach spaces $A_k$)

$$(l_{q_0}^0(A_0), l_{q_1}^0(A_1))_{s, q} = l_{q}^0((A_0, A_1)_{s, q}), \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}, \quad s = (1 - \theta) s_0 + \theta s_1,$$

with $0 < q_0, q_1 < \infty$, $0 < \theta < 1$ and $A_i := H_{p_i}^0$ ($i = 0, 1$), and the interpolation formula (11).

3. CLASSICAL ANISOTROPIC FUNCTION SPACES

Now we compare the spaces $B_{p,q}^s$ and $F_{p,q}^s$ with the classical anisotropic spaces.

If $s := (s_1, \ldots, s_n)$, $s_j > 0$, then the well-known anisotropic Lebesgue spaces (= Bessel-potential spaces) are defined by

$$
H_{p}^0 := \{ f \mid f \in S', \|f\|_{H_{p}^0} := \|F^{-1}(\sum_{j=1}^{n} (1 + x_j^{s_j})^{s_j/2} F f\|_{L_{p}} < \infty \}, \quad 1 < p < \infty.
$$

Denote as usual by $D_j := \partial / \partial x_j$, $D^x := D_{x_1}^{x_1} \ldots D_{x_n}^{x_n}$, $x := (x_1, \ldots, x_n)$, $x_j \geq 0$ integer, $0 < \alpha = (x_1, \ldots, x_n)$, $x_j \geq 0$ integer, the derivatives of functions or distributions on $R^n$. For $m := (m_1, \ldots, m_n)$, $m_j > 0$ integer, we have the anisotropic Sobolev spaces

$$
W_{p}^m := \{ f \mid f \in S', \|f\|_{W_{p}^m} := \sum_{\sum x_j / m_j \leq 1} \|D^x f\|_{L_{p}} < \infty \}, \quad 1 \leq p < \infty.
$$
and the anisotropic Hölder spaces
\[
C^m := \{ f \mid f \in C, \| f \|_{C^m} := \sum_{0 \leq \sum j/m_j \leq 1} \| D^s f \|_C < \infty \},
\]
where \( C \) is the set of all bounded uniformly continuous functions on \( \mathbb{R}^n \) and
\[
\| f \|_C := \sup_{x \in \mathbb{R}^n} |f(x)|.
\]

The following theorem shows the connection between the spaces \( F^s_{p,2} = H^s_p \) and the anisotropic Lebesgue- and Sobolev spaces.

**Theorem 2.** ([9], p. 266). If \( 1 < p < \infty \) and \( s := (s_1, \ldots, s_n), s_j > 0, \) then
\[
H^s_p = \delta^s_p,
\]
and, provided \( s_j (j = 1, \ldots, n) \) are integers,
\[
H^s_p = W^s_p,
\]
where the corresponding norms are equivalent.

Before we give a classical interpretation for the spaces \( B^s_{p,q} (1 \leq p, q \leq \infty, s > 0) \) we need the following proposition.

**Proposition 2.** Let \( a := (a_1, \ldots, a_n), a_j > 0, s := (s/a_1, \ldots, s/a_n), -\infty < s < \infty, \) be the numbers which characterize the anisotropic smoothness. Then
\[
B^0_{p,1} \subset L_p \subset B^0_{p,\infty}, \quad 1 \leq p \leq \infty,
\]
\[
B^0_{\infty,1} \subset C \subset B^0_{\infty,\infty},
\]
and if \( s/a_j = m_j (j = 1, \ldots, n) \) are integers,
\[
B^m_{p,1} \subset W^m_p \subset B^m_{p,\infty}, \quad 1 \leq p < \infty,
\]
\[
B^m_{\infty,1} \subset C^m \subset B^m_{\infty,\infty}.
\]

**Proof.** The proof of the proposition is the same as in the isotropic case (cf. [13], p. 68), if we use the following identity that holds for \( 0 < p \leq \infty, 0 < q \leq \infty \) and \( s/a_j = m_j (j = 1, \ldots, n) \) integers:
\[
B_{p,q}^m = \{ f \mid f \in S', \| f \|_{B_{p,q}^m} := \sum_{0 \leq \sum j/m_j \leq 1} \| D^s f \|_{B_{p,q}^m} < \infty \}.
\]
This statement follows from the lifting property for \( B_{p,q}^m \) (analogous to \( F^s_{p,q} \) \( p < \infty \)) [9], p. 265) by standard arguments (cf. [13], p. 67) using the anisotropic multiplier theorems ([9], p. 264).

**Theorem 3.** Let \( a := (a_1, \ldots, a_n), a_j > 0, s > 0, s := (s/a_1, \ldots, s/a_n) = (s_1, \ldots, s_n) \) be the multiindices which characterize the anisotropic smoothness and \( \delta := (\delta_1, \ldots, \delta_n), 0 < \delta_j \leq \infty, \beta := (\beta_1, \ldots, \beta_n), l := (l_1, \ldots, l_n), 0 \leq \beta_j, l_j \) integers...
(j = 1, ..., n), such that

\[ 0 \leq \beta_j < s_j, \quad s_j - \beta_j < l_j \quad (j = 1, ..., n), \]

then

\[ B^s_{p,q} = \mathcal{B}^s_{p,q}, \]

holds for \( 1 \leq p < \infty, \ 1 \leq q \leq \infty, \) where

\[ \mathcal{B}^s_{p,q} := \{ f \mid f \in L_p, \ \| f \|_{L^p,1,q,s} < \infty \}, \]

and the expressions

\[ \| f \|_{L,p,1,q,s} := \| f \|_{L^p} + \sum_{j=1}^{n} \left( \int_{0}^{\infty} \left( \int_{0}^{t} \| A^{l_j}_j f \|_{L^p} \right)^{1/q} \right) \]

are equivalent norms in \( \mathcal{B}^s_{p,q} \) and \( B^s_{p,q} \) for all admissible values of parameters.

For \( p = q = \infty \) we have

\[ B^s_{\infty,\infty} = \mathcal{B}^s_{\infty,\infty}, \]

where

\[ \mathcal{B}^s_{\infty,\infty} := \{ f \mid f \in C, \ \| f \|_{C,1,q,s} < \infty \} \]

and the expressions

\[ \| f \|_{C,p,1,q,s} := \| f \|_{C} + \sum_{j=1}^{n} \sup_{0 < t < \delta_j} \left( \int_{0}^{t} \| A^{l_j}_j f \|_{C} \right) \]

are equivalent norms in \( \mathcal{B}^s_{\infty,\infty} \) and \( B^s_{\infty,\infty} \) for all admissible values of parameters.

**Proof.** The theorem is based on interpolation theorems for classical anisotropic spaces and on Theorem 1.

For \( 1 \leq p < \infty, \ 0 < \theta < 1, \ 1 \leq q \leq \infty, \) H.-J. Schmeisser and H. Triebel have proved in [6] (cf. also [7], [8]) the identities

\[ (L^p, W^1_p)_{\theta,q} = \mathcal{B}^s_{p,q} \]

and

\[ (C, C^1)_{\theta,\infty} = \mathcal{B}^s_{\theta,\infty}. \]

We remark that the space \( C \) (completion of \( C^\infty_0(\mathbb{R}^n) \)) in [6], p. 120, can be replaced by our space \( C \).

Now we obtain the statements (32) and (34) from our Theorem 1, the formulas (28)–(31), (36), (37) and the reiteration theorem (cf. [1], p. 50).

**Remark 6.** Theorem 3 shows that our anisotropic \( B^s_{p,q} \) spaces for \( p, q \geq 1, s > 0 \) coincide with the classical ones.

(i) If \( s_j > 0 \) are arbitrary numbers and \( s_j = [s_j]^- + \{ s_j \}^- \) \( ([s_j]^- \) integer, 0 <
< \{ s_j \}^{-} \leq 1), then (33) and (35) for \( \beta_j := \lfloor s_j \rfloor \), \( s_j - \beta_j := \{ s_j \} \), \( l_j := 2 \), \( \delta_j := \infty \) \( (j = 1, \ldots, n) \) give the well-known norms for the classical anisotropic Besov spaces \( \mathcal{B}^s_{p,q} \) and Zygmund spaces \( \mathcal{C}^s \), respectively (cf. [3], [4]) and we get from (34) and (36):
\[
\mathcal{B}^s_{p,q} = \mathcal{B}^s_{p,q}, \quad \mathcal{B}^s_{\infty,\infty} = \mathcal{C}^s,
\]
\( (1 \leq p < \infty, 1 \leq q \leq \infty, s = (s_1, \ldots, s_n), s_j > 0). \)

(ii) If \( s_j > 0 \) is not an integer and \( s_j = \lfloor s_j \rfloor + \{ s_j \} \) integer, \( 0 \leq \lfloor s_j \rfloor < 1 \), then (33) and (35) for \( \beta_j := \lfloor s_j \rfloor \), \( s_j - \beta_j := \{ s_j \} \), \( l_j := 1 \), \( \delta_j := \infty \) \( (j = 1, \ldots, n) \) give the well-known norms for the classical anisotropic Slobodeckij spaces \( \mathcal{W}^s_p \) \( (p = q) \) and Hölder spaces \( \mathcal{C}^s \), respectively, and (34), (36) imply
\[
\mathcal{B}^s_{p,p} = \mathcal{W}^s_p, \quad \mathcal{B}^s_{\infty,\infty} = \mathcal{C}^s,
\]
\( (1 \leq p < \infty, s = (s_1, \ldots, s_n), 0 < s_j \neq \text{integer}). \)

**Remark 7.** Together with the classical interpretations of the spaces \( \mathcal{B}^s_{p,q} \) and \( \mathcal{W}^s_p \), Theorem 1 yields new interpolation formulas, e.g. for \( a := (a_1, \ldots, a_n) \), \( a_j > 0 \);
\( s_i > 0, s_i := (s_i/a_1, \ldots, s_i/a_n), i = 0, 1; 1 \leq p_0 < p_1 < \infty, 1 \leq q_0; q_1 < \infty \) we get
\[
(\mathcal{B}_{p_0,q_0}^{s_0,\ldots,s_{i-1},s_i}, \mathcal{B}_{p_1,q_1}^{s_0,\ldots,s_{i-1},s_i})_{\theta,p} = \mathcal{B}_{p,p}^{s}, \quad s = (1 - \theta) s_0 + \theta s_1,
\]
\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}
\]
and
\[
(L_{p_0}, \mathcal{B}_{p_1,q_1}^{s_0,\ldots,s_{i-1},s_i})_{\theta,p} = \mathcal{B}_{p,p}^{s}, \quad s = \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},
\]
which complete the interpolation results on the classical anisotropic function spaces in [11], [7].

**References**


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