Konstantin Igorevich Beidar
A chain of Kurosh may have an arbitrary finite length


Persistent URL: [http://dml.cz/dmlcz/101818](http://dml.cz/dmlcz/101818)

Terms of use:

© Institute of Mathematics AS CR, 1982

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://dml.cz](http://dml.cz)
A CHAIN OF KUROSH MAY HAVE AN ARBITRARY FINITE LENGTH

K. I. BEIDAR, Moscow

(Received June 4, 1981)

Let $M$ be any nonempty homomorphically closed class of associative rings. The chain of Kurosh of the class $M$ is the chain of classes $M = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_t \subseteq \ldots$ with $t$ running over all ordinal numbers, where the class $M_t$ consists of all associative rings whose every nonzero homomorphic image contains a nonzero ideal which is in the class $\bigcup_{r \leq t} M_r$ [1, p. 113, Definition 2].

Sulinski, Anderson and Divinsky [5] have shown that this chain of classes terminates at $\omega_0$, the first infinite ordinal number, and also have constructed examples of homomorphically closed classes of rings whose chains of Kurosh terminate at the second or the third step. They posed the following problem: Is it possible, for every natural number $n$, to construct a homomorphically closed class $M^{(n)}$ whose chain of Kurosh terminates precisely at the step $n$?

In the present note such classes will be constructed. Let $C$ be the field of complex numbers, $Q$ — the subfield of rational numbers, $Z$ — the subring of integers, $D = Q[i]$ — the subfield of the field $C$ generated by $Q$, and $i$ — the square root from $-1$. Further, let $p$ be a prime number of a form $4s + 3$ and

$$A_n = pZ + ip^nZ \subseteq D, \quad A_0 = Z[i] \subseteq D, \quad n = 1, 2, \ldots$$

It is clear that $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n \supseteq \ldots$ and $A_{n+1}$ is an ideal of the ring $A_n$, $n = 0, 1, \ldots$.

**Lemma 1.** Let $L$ be a nonzero ideal of the ring $A_n$, $R$ — the subring of the field $D$ such that $A_n$ is an ideal of the ring $R$, let $f : A_n \rightarrow D$ be a homomorphism of the rings. Then: 1. $A_n/L$ is a finite ring. 2. Either $R = A_{n-1}$, or $R = A_n$, or $R \ni 1$. 3. Either $f(A_n) = 0$, or $f(A_n) = A_n$. 4. If $A_{n+m}$ is an ideal of the ring $A_n$, then $0 \leq m \leq 1$.

**Proof.** 1. Let $0 \neq m + in \in L$, where $m, n \in Z$. Then

$$a = m^2 + n^2 = (m + in)(m - in) \in L.$$

It is clear that $A_n/L$ is a module with two generators $ip^n + L$ and $p + L$ over the finite ring $Z/aZ$. Hence $A_n/L$ is a finite ring.
2. Since \( A_0 \ni 1 \), the case \( n = 0 \) is evident. Let \( n > 0 \). Then \( R \subseteq A_0 \), because \( RA_n \subseteq A_n \). Hence all \( x \in R \) have a form \( x = m + in \), where \( m = m(x) \), \( n = n(x) \in Z \). Now assume that \( n = 1 \). Then \( A_1 = pA_0 \) and

\[
R/A_1 \subseteq A_0/A_1 = (Z/pZ)[i].
\]

The square root from \(-1\) is not contained in the field \( Z/pZ \), because the prime number \( p \) has a form \( 4s + 3 \) [3, p. 68]. Therefore \((Z/pZ)[i]\) is a field. A nonzero subring of a finite field must be a field. Hence either \( R/A_1 = 0 \) and \( R = A_1 \), or \( R/A_1 \ni 1 + A_1 \). Let \( 1 + x = y \) for some \( x \in A_1 \), \( y \in R \). Therefore \( 1 = y - x \in R \) and our statement is proved. Let now \( n \geq 2 \). Since \( p \in A_n \), \( px \in A_n \) for all \( x \in R \). By definition of \( A_n \) it follows that \( n(x) = p^{n-1}l(x) \), where \( l(x) \in Z \). Hence \( x = m + ip^{n-1}l \). Let us suppose that \( pZ + m(x)Z = Z \) for some \( x \in R \). Then \( up + vm = 1 \) for some \( u, v \in Z \). Define \( z = up + vx \). It is clear that \( z \in R \) and \( z = 1 + ip^{n-1}s \), where \( s = vl \). Since \( n \geq 2 \), \( 2n - 3 > 0 \) and \( p^{2n-2}s^2 = = p \cdot p^{n-3}s^2 \in R \) (recall that \( p \in A_n \subseteq R \)). Therefore \( 1 = z - z^2 - p^{2n-2}s^2 \in R \).

So we may assume that \( m(x) = p(s(x)) \), where \( s(x) \in Z \) for all \( x \in R \). Since \( p \in A_n \), it follows that the additive group \( R \) is generated by the subgroup \( A_n \) and the set \( \{ip^{n-1}l(x) = x - p(s(x))x \in R \} \). If \( p \) divides \( l(x) \) for all \( x \in R \), then \( ip^{n-1}l(x) \in pZ + ip^nZ = A_n \) and \( R = A_n \). Let us suppose that \( pZ + l(x)Z = Z \) for some \( x \in R \). Then \( ap + bl = 1 \) for some \( a, b \in Z \). Since \( ip^n \in A_n \), we have \( ip^{n-1} = ip^{n-1}(ap + + bl) = ip^na + ip^nbl \in R \) and in this case \( R = A_{n-1} \).

3. Since the field \( D \) does not contain a finite nonzero subring, it follows that either \( f(A_n) = 0 \), or \( f \) is a monomorphism. Let \( S = Z \setminus \{0\} \) and let \( S^{-1}A_n \) be the ring of fractions of the \( Z \)-algebra \( A_n \) with respect to \( S \) [2, p. 49]. It is clear that \( S^{-1}A_n = D \) and the monomorphism \( f \) may be continued to the monomorphism of the ring \( f: S^{-1}A_n \to D \). Further, it is obvious that the field \( D \) has only two monomorphisms: the identical one and the complex conjugation. Hence \( f(A_n) = A_n \).

4. It is clear that \( p \in A_{n+m} \) and \( ip^n \in A_n \). Therefore \( ip^{n+1} = p \cdot ip^n \in A_{n+m} = = pZ + ip^{n+m}Z \) and \( n + 1 \geq n + m \). Hence \( m \leq 1 \).

**Lemma 2.** Let \( M^{(n+1)} \) be the homomorphically closed class of rings, which consists of all nilpotent rings, all finite commutative rings and all homomorphic images of the ring \( A_n \). Then \( A_{n-m} \in M^{(n+1)} \setminus M^{(n+1)} \) for all \( m = 1, 2, \ldots, n \) and \( n > 0 \).

**Proof.** By Lemma 1, each homomorphic image of the ring \( A_{n-m} \) with a nonzero kernel is a finite commutative ring. Hence all such homomorphic images of the ring \( A_{n-m} \) are contained in the class \( M^{(n+1)} \).

Let \( m = 1 \). It is clear that \( A_n \in M^{(n+1)} \) and \( A_n \) is an ideal of the ring \( A_{n-1} \). It follows that \( A_{n-1} \in M^{(n+1)} \). Suppose \( A_{n-1} \in M^{(n+1)} \). By the definition of the class \( M^{(n+1)} \) it follows that the ring \( A_{n-1} \) is isomorphic to the ring \( A_n \). Let \( f: A_{n-1} \to A_n \) be an isomorphism. Since \( A_n \subseteq D \), we have \( f(A_{n-1}) = A_{n-1} \neq A_n \) (see Lemma 1). We
obtain a contradiction. Hence

\[ A_{n-1} \in M_{2}^{(n+1)} \setminus M_{1}^{(n+1)}. \]

Now we proceed by induction on \( m \). The case \( m = 1 \) has been proved. Assume that the lemma is true for \( l < m \). Then \( A_{n-m+1} \in M_{m}^{(n+1)} \). Since \( A_{n-m+1} \) is an ideal of the ring \( A_{n-m} \), it follows that \( A_{n-m} \in M_{m}^{(n+1)} \). Suppose now that \( A_{n-m} \in M_{m}^{(n+1)} \). Then the ring \( A_{n-m} \) contains a nonzero ideal \( B_{1} \) from the class \( M_{m}^{(n+1)} \). The ring \( B_{1} \) also contains a nonzero ideal \( B_{2} \) of the class \( M_{m-2}^{(n+1)} \). Continuing this process we obtain a chain of nonzero subrings \( B_{m-1} \subseteq B_{m-2} \subseteq \ldots \subseteq B_{1} \subseteq A_{n-m} \), where \( B_{m-i} \) is an ideal of the ring \( B_{m-i-1} \) and \( B_{m-i} \in M_{m}^{(n+1)} \) for all \( i = 1, 2, \ldots, m-1 \). Since the ring \( A_{n-m} \) contains neither finite nor nilpotent nonzero subrings, the ring \( B_{m-1} \) is isomorphic to the ring \( A_{n} \). As above, we obtain that \( B_{m-1} = A_{n} \).

Consider now the case \( B_{m-i} \neq 1, i = 1, 2, \ldots, m-1 \). By Lemma 1, our chain has a form

\[ A_{n} \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-t} \subseteq A_{n-m}, \]

where \( t \leq m-2 \) and \( A_{n-t} = B_{1} \) is an ideal of the ring \( A_{n-m} \). By Lemma 1 (assertion 4), \( t \geq m-1 \). We obtain a contradiction. So the ring \( B_{m-i} \) must contain identity for some \( 1 \leq i \leq m-1 \).

Suppose now that the ring \( B_{m-i} \) contains an identity for some \( 1 \leq i \leq m-1 \). Since \( B_{m-1} = A_{n} \neq 1 \), we have \( i > 1 \). Hence we can assume that the ring \( B_{m-i+1} \) does not contain identity. It is clear that

\[ B_{m-i} = B_{m-i-1} = \ldots = B_{1} = A_{n-m}, \quad A_{n-m} = A_{0} \]

and \( m = n \). By Lemma 1 our chain has the form \( A_{n} \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-t} \subseteq A_{0} \), where \( t \leq i - 1 \leq m - 2 = n - 2 \) and \( A_{n-t} = B_{m-i+1} \) is an ideal of the ring \( B_{m-i} = A_{0} \). By Lemma 1 (assertion 4), \( t \geq n-1 \). But \( t \leq n-2 \). We obtain a contradiction. So \( A_{n-m} \in M_{m}^{(n+1)} \setminus M_{m}^{(n+1)} \) for all \( 1 \leq m \leq n \) and \( n > 0 \). This completes the proof of the lemma.

**Corollary 3.** Let \( n > 0 \), let \( A_{n} = B_{1} \subseteq B_{2} \subseteq \ldots \subseteq B_{m} \subseteq D \) be a chain of subrings of the field \( D \). Assume that \( B_{1} \) is an ideal of the ring \( B_{i+1} \) for all \( i = 1, 2, \ldots, m-1 \). Then: 1. If \( B_{m} \neq 1 \), then our chain of subrings has a form \( A_{n} \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-t} \subseteq B_{m} \). 2. If \( B_{1} \neq 1 \) and \( B_{i-1} \neq 1 \) for some \( i \geq 2 \), then our chain of subrings has a form

\[ A_{n} \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-t} \subseteq B_{i} = B_{i+1} = \ldots = B_{m}, \]

where \( t \leq i - 2 \) and \( A_{n-t} = B_{i-1} \) is an ideal of the ring \( B_{i} \).

**Lemma 4.** Let \( B_{i} \subseteq B_{i-1} \subseteq \ldots \subseteq B_{0} \) be a chain of rings such that \( B_{1} \) is an ideal of the ring \( B_{i-1} \) for all \( 1 \leq i \leq t \). If \( x \in B_{i} \), then \( x^{m} y \in B_{m} \). If \( x \in B_{i} \), then \( x^{m} y \in B_{m} \).
for all \( m = 1, 2, \ldots, t. \) If \( e \) is a central idempotent of the ring \( B_i \), then \( e \) is a central idempotent of the ring \( B_0 \) and \( eB_i = eB_0 \).

**Proof.** 1. We shall prove that \( P^{3m} \subseteq B_mB_mB_m \) for all \( 1 \leq m \leq t. \) Indeed, \( B_i \subseteq B_1 \) and \( B_1 \) is an ideal of the ring \( B_0. \) Hence \( P \subseteq B_1 \) and \( P^3 \subseteq B_1B_1B_1. \) We proceed by induction on \( m. \) Assume that \( B_mB_iB_m \cong P^{3m} \). Since \( B_{m+1} \) is an ideal of the ring \( B_m \) and \( B_i \subseteq B_{m+1}, \) we have

\[
P^{3m} \subseteq B_mB_mB_m \subseteq B_{m+1},
\]

\[
P^{3m+1} = P^{3m}P^{3m}P^{3m} \subseteq P^{3m}B_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_mB_MB

2. We proceed by induction on \( m. \) Since \( x \in B_i \subseteq B_1 \) and \( B_1 \) is an ideal of the ring \( B_0, xy \in B_1. \) Assume that \( x^m y \in B_n. \) We have \( x \in B_i \subseteq B_{m+1} \), and \( B_{m+1} \) is an ideal of the ring \( B_m. \) Therefore \( x^{m+1}y = x(xy) \in B_{m+1}. \)

3. Since \( e^t = e, ex \in B_i \) and \( xe \in B_i \) for all \( x \in B_0. \) But \( e \) is a central idempotent of the ring \( B_i. \) Hence \( ex = e(ex) = (ex)e = e(xe) = xe \) for all \( x \in B_0. \) Therefore \( e \) is a central idempotent of the ring \( B_0. \) It is clear that \( eB_0 \subseteq B_i \subseteq B_0. \) Hence \( eB_0 = (eB_0) \subseteq eB_i \subseteq eB_0 \) and \( eB_i = eB_0. \)

**Theorem 5.** Let \( Z \) be the ring of integers, \( i \) the square root of \( -1, p \) a prime number of the form \( p = 4s + 3, \) \( A_n = pZ + ip^nZ \) and \( M^{(n+1)}_1 \) a homomorphically closed class of rings, which consists of all nilpotent rings, all homomorphic images of the ring \( A_n \) and all finite commutative rings. Then the chain of Kurosh of the class \( M^{(n+1)}_1 \) terminates precisely at the step \( n + 1. \)

**Proof.** By Lemma 2, \( M^{(n+1)}_{n+2} \neq M^{(n+1)}_n. \) Hence it suffices to prove that \( M^{(n+1)}_{n+1} = M^{(n+1)}_n. \) Since \( M^{(n+1)}_{n+1} \) is a homomorphically closed class of rings, it suffices to prove that

\( (*) \) each nonzero ring of the class \( M^{(n+1)}_m \) contains a nonzero ideal of the class \( M^{(n+1)}_n. \) Let \( 0 \neq B \in M^{(n+1)}_n. \) Since \( M^{(n+1)}_1 \subseteq M^{(n+1)}_n, \) all nilpotent rings are contained in the class \( M^{(n+1)}_n. \) Therefore we can assume that \( B \) is a semiprime ring. The ring \( B \) contains such a chain of nonzero subrings

\[
B_i \subseteq B_{i-1} \subseteq \ldots \subseteq B_0 = B
\]

that \( B_i \in M^{(n+1)}_m \) and \( B_i \) is an ideal of the ring \( B_{i-1} \) for all \( i = 1, 2, \ldots, t. \) [5, p. 418, Lemma 1]. An ideal of a semiprime ring is itself a semiprime ring. Hence \( B_i \) is a semiprime ring. Thus there are only two possibilities: a) \( B_i \) is a finite commutative ring; b) the ring \( B_i \) is isomorphic to the ring \( A_n. \)

Let us consider the first case. It is clear that \( B_i \) is an artinian ring. Since \( B_i \) is a semiprime ring, it has an identity \( e. \) By Lemma 4, \( Be = B_i e = B_i. \) Hence \( B_i \) is an ideal of the ring \( B. \) Since \( B_i \in M^{(n+1)}_1 \subseteq M^{(n+1)}_n, \) the statement \( (*) \) is proved.

Now let us consider the second case. We can assume that \( B_i = A_n. \) Let

\[
P = B_t + BB_t + B,B + BB,B, \quad r(B; P) = \{ b \in B | Pb = 0 \}.
\]

421
By Lemma 4, \( P^m \subseteq B_i \subseteq P \) for some \( m \). Hence we have

\[
K = r(B; P^m) \supseteq r(B; B_i) \supseteq r(B; P).
\]

Further, \( (PK)^n = PKPK \ldots PK \subseteq P^mK = 0 \). Since \( B \) is a semiprime ring, \( PK = 0 \) and \( K \subseteq r(B; P) \). Hence \( K = r(B; B_i) = r(B; P) \) (see (**)). Therefore \( r(B; B_i) \) is an ideal of the ring \( B \). Let \( L = \{ b \in B | bK = 0 \} \). It is clear that \( L \) is an ideal of the ring \( B \) and \( L \cap K = 0 \). Let \( H_i = B_i \cap L \) for all \( i = 1, 2, \ldots, t \). Evidently, \( H_i = B_i \).

\( H_i \) is an ideal of the ring \( H_{i-1} \) for all \( 1 \leq i \leq t \) and \( H_0 = L \) is an ideal of the ring \( B \).

Further, \( r(H_0; B_i) = K \cap L = 0 \). Assume that \( na = 0 \) for some \( 0 \neq n \in Z \) and \( 0 \neq a \in H_0 \). Since \( B_0 a \neq 0 \) and \( B_0 an = 0 \), \( T = \{ b \in P | bn = 0 \} \neq 0 \). It is clear that \( T \) is an ideal of the ring \( B \) and \( T^m \subseteq P^m \subseteq B_i \).

Since \( xn \neq 0 \) for all \( 0 \neq x \in B_i = A_n \), \( T^m = 0 \). But \( B \) is a semiprime ring. We obtain a contradiction. Therefore \( na \neq 0 \) for all \( 0 \neq a \in H_0 \) and \( 0 \neq n \in Z \). Let \( S = Z \setminus \{0\} \) and let \( S^{-1}H_0 \) be a localization of the ring \( H_0 \) (see [2]). By the above, \( H_0 \subseteq S^{-1}H_0 \) and

\[
r(S^{-1}H_0; S^{-1}B_i) = S^{-1}r(H_0; B_i) = 0.
\]

It is clear that \( S^{-1}B_i = S^{-1}A_n = D \) and the identity of the field \( D \) will be the identity for all rings \( S^{-1}H_i, i = 0, 1, \ldots, t \). Evidently, \( S^{-1}H_i \) is an ideal of the ring \( S^{-1}H_{i-1} \).

\[
D = S^{-1}H_t = S^{-1}H_{t-1} = \ldots = S^{-1}H_0, \quad H_0 \subseteq D.
\]

Assume now that the ring \( H_0 \) does not contain identity. By Corollary 3, \( H_0 = A_{n-r} \) for some \( r \leq t \). Since \( A_0 \equiv 1, n - r \neq 0, r < n \). Further, by Lemma 2, \( A_{n-r} \in M_{r+1} \subseteq M_{n+1} \). Therefore in this case the condition (*) holds.

Suppose now that the ring \( H_0 \) contains an identity \( e \). By Corollary 3, the chain of subrings \( A_n = H_i \subset H_{i-1} \subset \ldots \subset H_0 \) has a form \( A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-r} \subseteq H_i = \ldots = H_{i-r} = \ldots = H_0 \) and \( A_{n-r} \) is an ideal of the ring \( H_0 \) and \( n - r \neq 0 \). By Lemma 4, \( eB = eH_0 = H_0 \).

Since \( A_{n-r}e = A_{n-r}, A_{n-r}B = A_{n-r}H_0 \subseteq A_{n-r} \),

Similarly \( BA_{n-r} \subseteq A_{n-r} \).

Hence \( A_{n-r} \) is an ideal of the ring \( B \). It is clear that \( A_{n-r} \subseteq \subseteq M_{n+1} \subseteq M_{n+1} \) (see Lemma 2). Therefore the condition (*) holds in all cases. This completes our proof.

References


Author’s address: Department of Mathematics and Mechanics, Moscow State University, Moscow, 117234, USSR.

422