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A CHAIN OF KUROSH MAY HAVE AN ARBITRARY FINITE LENGTH

K. I. BEIDAR, Moscow

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Let M be any nonempty homomorphically closed class of associative rings. The chain of Kurosh of the class M is the chain of classes $M = M_0 \subseteq M_1 \subseteq \ldots \ldots \subseteq M_t \subseteq \ldots$ with t running over all ordinal numbers, where the class M_t consists of all associative rings whose every nonzero homomorphic image contains a nonzero ideal which is in the class $\bigcup_{r<t} M_r$ [1, p. 113, Definition 2].

Sulinski, Anderson and Divinsky [5] have shown that this chain of classes terminates at ω_0, the first infinite ordinal number, and also have constructed examples of homomorphically closed classes of rings whose chains of Kurosh terminate at the second or the third step. They posed the following problem: Is it possible, for every natural number n, to construct a homomorphically closed class $M^{(n)}$ whose chain of Kurosh terminates precisely at the step n?

In the present note such classes will be constructed. Let C be the field of complex numbers, Q – the subfield of rational numbers, Z – the subring of integers, $D = Q[i]$ – the subfield of the field C generated by Q, and i – the square root from -1. Further, let p be a prime number of a form $4s + 3$ and

$$A_n = pZ + \mathrm{i}p^n Z \subseteq D\,, \quad A_0 = Z[\mathrm{i}] \subseteq D\,, \quad n = 1, 2, \ldots .$$

It is clear that $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n \supseteq \ldots$ and A_{n+1} is an ideal of the ring A_n, $n = 0, 1, \ldots$.

Lemma 1. *Let L be a nonzero ideal of the ring A_n, R – the subring of the field D such that A_n is an ideal of the ring R, let $f : A_n \to D$ be a homomorphism of the rings. Then:* 1. *A_n/L is a finite ring.* 2. *Either $R = A_{n-1}$, or $R = A_n$, or $R \ni 1$.* 3. *Either $f(A_n) = 0$, or $f(A_n) = A_n$.* 4. *If A_{n+m} is an ideal of the ring A_n, then $0 \leqq m \leqq 1$.*

Proof. 1. Let $0 \neq m + \mathrm{i}n \in L$, where $m, n \in Z$. Then

$$a = m^2 + n^2 = (m + \mathrm{i}n)(m - \mathrm{i}n) \in L\,.$$

It is clear that A_n/L is a module with two generators $\mathrm{i}p^n + L$ and $p + L$ over the finite ring Z/aZ. Hence A_n/L is a finite ring.

2. Since $A_0 \ni 1$, the case $n = 0$ is evident. Let $n > 0$. Then $R \subseteq A_0$, because $RA_n \subseteq A_n$. Hence all $x \in R$ have a form $x = m + in$, where $m = m(x)$, $n = n(x) \in \in Z$. Now assume that $n = 1$. Then $A_1 = pA_0$ and

$$R/A_1 \subseteq A_0/A_1 = (Z/pZ)[\mathrm{i}].$$

The square root from -1 is not contained in the field Z/pZ, because the prime number p has a form $4s + 3$ [3, p. 68]. Therefore $(Z/pZ)[\mathrm{i}]$ is a field. A nonzero subring of a finite field must be a field. Hence either $R/A_1 = 0$ and $R = A_1$, or $R/A_1 \ni 1 + A_1$. Let $R/A_1 \ni 1 + A_1$. Then $1 + x = y$ for some $x \in A_1$, $y \in R$. Therefore $1 = y - x \in R$ and our statement is proved. Let now $n \geqq 2$. Since $p \in A_n$, $px \in A_n$ for all $x \in R$. By definition of A_n it follows that $n(x) = p^{n-1}\, l(x)$, where $l(x) \in Z$. Hence $x = m + \mathrm{i}p^{n-1}l$. Let us suppose that $pZ + m(x)\,Z = Z$ for some $x \in R$. Then $up + vm = 1$ for some $u, v \in Z$. Define $z = up + vx$. It is clear that $z \in R$ and $z = 1 + \mathrm{i}p^{n-1}s$, where $s = vl$. Since $n \geqq 2$, $2n - 3 > 0$ and $p^{2n-2}s^2 = = p \cdot p^{2n-3}s^2 \in R$ (recall that $p \in A_n \subseteq R$). Therefore $1 = 2z - z^2 - p^{2n-2}s^2 \in R$. So we may assume that $m(x) = p\, s(x)$, where $s(x) \in Z$ for all $x \in R$. Since $p \in A_n$, it follows that the additive group R is generated by the subgroup A_n and the set $\{\mathrm{i}p^{n-1}\, l(x) = x - p\, s(x) / x \in R\}$. If p divides $l(x)$ for all $x \in R$, then $\mathrm{i}p^{n-1}\, l(x) \in \in pZ + \mathrm{i}p^n Z = A_n$ and $R = A_n$. Let us suppose that $pZ + l(x)\,Z = Z$ for some $x \in R$. Then $ap + bl = 1$ for some $a, b \in Z$. Since $\mathrm{i}p^n \in A_n$, we have $\mathrm{i}p^{n-1} = \mathrm{i}p^{n-1}(ap + + bl) = \mathrm{i}p^n a + \mathrm{i}p^{n-1}lb \in R$ and in this case $R = A_{n-1}$.

3. Since the field D does not contain a finite nonzero subring, it follows that either $f(A_n) = 0$, or f is a monomorphism. Let $S = Z \setminus \{0\}$ and let $S^{-1}A_n$ be the ring of fractions of the Z-algebra A_n with respect to S [2, p. 49]. It is clear that $S^{-1}A_n = D$ and the monomorphism f may be continued to the monomorphism of the ring $f\colon S^{-1}A_n \to D$. Further, it is obvious that the field D has only two monomorphisms: the identical one and the complex conjugation. Hence $f(A_n) = A_n$.

4. It is clear that $p \in A_{n+m}$ and $\mathrm{i}p^n \in A_n$. Therefore $\mathrm{i}p^{n+1} = p \,.\, \mathrm{i}p^n \in A_{n+m} = = pZ + \mathrm{i}p^{n+m}Z$ and $n + 1 \geqq n + m$. Hence $m \leqq 1$.

Lemma 2. *Let $M_1^{(n+1)}$ be the homomorphically closed class of rings, which consists of all nilpotent rings, all finite commutative rings and all homomorphic images of the ring A_n. Then $A_{n-m} \in M_{m+1}^{(n+1)} \setminus M_m^{(n+1)}$ for all $m = 1, 2, \ldots, n$ and $n > 0$.*

Proof. By Lemma 1, each homomorphic image of the ring A_{n-m} with a nonzero kernel is a finite commutative ring. Hence all such homomorphic images of the ring A_{n-m} are contained in the class $M_1^{(n+1)}$.

Let $m = 1$. It is clear that $A_n \in M_1^{(n+1)}$ and A_n is an ideal of the ring A_{n-1}. It follows that $A_{n-1} \in M_2^{(n+1)}$. Suppose $A_{n-1} \in M_1^{(n+1)}$. By the definition of the class $M_1^{(n+1)}$ it follows that the ring A_{n-1} is isomorphic to the ring A_n. Let $f : A_{n-1} \to A_n$ be an isomorphism. Since $A_n \subseteq D$, we have $f(A_{n-1}) = A_{n-1} \neq A_n$ (see Lemma 1). We obtain a contradiction. Hence

$$A_{n-1} \in M_2^{(n+1)} \setminus M_1^{(n+1)} .$$

Now we proceed by induction on m. The case $m = 1$ has been proved. Assume that the lemma is true for $l < m$. Then $A_{n-m+1} \in M_m^{(n+1)}$. Since A_{n-m+1} is an ideal of the ring A_{n-m}, it follows that $A_{n-m} \in M_{m+1}^{(n+1)}$. Suppose now that $A_{n-m} \in M_m^{(n+1)}$. Then the ring A_{n-m} contains a nonzero ideal B_1 from the class $M_{m-1}^{(n+1)}$. The ring B_1 also contains a nonzero ideal B_2 of the class $M_{m-2}^{(n+1)}$. Continuing this process we obtain a chain of nonzero subrings $B_{m-1} \subseteq B_{m-2} \subseteq \ldots \subseteq B_1 \subseteq A_{n-m}$, where B_{m-i} is an ideal of the ring B_{m-i-1} and $B_{m-i} \in M_i^{(n+1)}$ for all $i = 1, 2, \ldots, m-1$. Since the ring A_{n-m} contains neither finite nor nilpotent nonzero subrings, the ring B_{m-1} is isomorphic to the ring A_n. As above, we obtain that $B_{m-1} = A_n$.

Consider now the case $B_{m-i} \not\ni 1$, $i = 1, 2, \ldots, m-1$. By Lemma 1, our chain has a form

$$A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-t} \subseteq A_{n-m} ,$$

where $t \leqq m-2$ and $A_{n-t} = B_1$ is an ideal of the ring A_{n-m}. By Lemma 1 (assertion 4), $t \geqq m-1$. We obtain a contradiction. So the ring B_{m-i} must contain identity for some $1 \leqq i \leqq m-1$.

Suppose now that the ring B_{m-i} contains an identity for some $1 \leqq i \leqq m-1$. Since $B_{m-1} = A_n \not\ni 1$, we have $i > 1$. Hence we can assume that the ring B_{m-i+1} does not contain identity. It is clear that

$$B_{m-i} = B_{m-i-1} = \ldots = B_1 = A_{n-m} , \quad A_{n-m} = A_0$$

and $m = n$. By Lemma 1 our chain has the form $A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-t} \subseteq A_0$, where $t \leqq i-1 \leqq m-2 = n-2$ and $A_{n-t} = B_{m-i+1}$ is an ideal of the ring $B_{m-i} = A_0$. By Lemma 1 (assertion 4), $t \geqq n-1$. But $t \leqq n-2$. We obtain a contradiction. So $A_{n-m} \in M_{m+1}^{(n+1)} \setminus M_m^{(n+1)}$ for all $1 \leqq m \leqq n$ and $n > 0$. This completes the proof of the lemma.

Corollary 3. *Let $n > 0$, let $A_n = B_1 \subseteq B_2 \subseteq \ldots \subseteqq B_m \subseteq D$ be a chain of subrings of the field D. Assume that B_i is an ideal of the ring B_{i+1} for all $i = 1, 2, \ldots, m-1$. Then*: 1. *If $B_m \not\ni 1$, then our chain of subrings has a form $A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-t}$, where $t \leqq m-1$ and $A_{n-t} = B_m$.* 2. *If $B_i \ni 1$ and $B_{i-1} \not\ni 1$ for some $i \geqq 2$, then our chain of subrings has a form*

$$A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-t} \subseteq B_i = B_{i+1} = \ldots = B_m ,$$

where $t \leqq i-2$ and $A_{n-t} = B_{i-1}$ is an ideal of the ring B_i.

Lemma 4. *Let $B_t \subseteq B_{t-1} \subseteq \ldots \subseteq B_0$ be a chain of rings such that B_i is an ideal of the ring B_{i-1} for all $1 \leqq i \leqq t$, $P = B_t + B_0B_t + B_tB_0 + B_0B_tB_0$. Then*: 1. *$P^{3^m} \subseteq B_m$ for all $m = 1, 2, \ldots, t$.* 2. *If $x \in B_t$, $y \in B_0$, then $x^m y \in B_m$, $yx^m \in B_m$*

for all $m = 1, 2, \ldots, t$. 3. *If* e *is a central idempotent of the ring* B_t, *then* e *is a central idempotent of the ring* B_0 *and* $eB_t = eB_0$.

Proof. 1. We shall prove that $P^{3^m} \subseteq B_m B_t B_m$ for all $1 \leqq m \leqq t$. Indeed, $B_t \subseteq B_1$ and B_1 is an ideal of the ring B_0. Hence $P \subseteq B_1$ and $P^3 \subseteq B_1 B_t B_1$. We proceed by induction on m. Assume that $B_m B_t B_m \supseteq P^{3^m}$. Since B_{m+1} is an ideal of the ring B_m and $B_t \subseteq B_{m+1}$, we have

$$P^{3^m} \subseteq B_m B_t B_m \subseteq B_{m+1}\,,$$

$$P^{3^{m+1}} = P^{3^m} P^{3^m} P^{3^m} \subseteq P^{3^m} B_m B_t B_m P^{3^m} \subseteq B_{m+1} B_m B_t B_m B_{m+1} \subseteq$$

$$\subseteq B_{m+1} B_t B_{m+1}\,.$$

2. We proceed by induction on m. Since $x \in B_t \subseteq B_1$ and B_1 is an ideal of the ring B_0, $xy \in B_1$. Assume that $x^m y \in B_m$. We have $x \in B_t \subseteq B_{m+1}$, and B_{m+1} is an ideal of the ring B_m. Therefore $x^{m+1} y = x(x^m y) \in B_{m+1}$.

3. Since $e^t = e$, $ex \in B_t$ and $xe \in B_t$ for all $x \in B_0$. But e is a central idempotent of the ring B_t. Hence $ex = e(ex) = (ex)\,e = e(xe) = (xe)\,e = xe$ for all $x \in B_0$. Therefore e is a central idempotent of the ring B_0. It is clear that $eB_0 \subseteq B_t \subseteq B_0$. Hence $eB_0 = e(eB_0) \subseteq eB_t \subseteq eB_0$ and $eB_t = eB_0$.

Theorem 5. *Let* Z *be the ring of integers,* i *the square root from* -1, p *a prime number of the form* $p = 4s + 3$, $A_n = pZ + \mathrm{i}p^n Z$ *and* $M_1^{(n+1)}$ *a homomorphically closed class of rings, which consists of all nilpotent rings, all homomorphic images of the ring* A_n *and all finite commutative rings. Then the chain of Kurosh of the class* $M_1^{(n+1)}$ *terminates precisely at the step* $n + 1$.

Proof. By Lemma 2, $M_{n+1}^{(n+1)} \neq M_n^{(n+1)}$. Hence it suffices to prove that $M_{n+1}^{(n+1)} = M_{n+2}^{(n+1)}$. Since $M_{n+2}^{(n+1)}$ is a homomorphically closed class of rings, it suffices to prove that

(*) each nonzero ring of the class $M_{n+2}^{(n+1)}$ contains a nonzero ideal of the class $M_n^{(n+1)}$.

Let $0 \neq B \in M_{n+2}^{(n+1)}$. Since $M_1^{(n+1)} \subseteq M_n^{(n+1)}$, all nilpotent rings are contained in the class $M_n^{(n+1)}$. Therefore we can assume that B is a semiprime ring. The ring B contains such a chain of nonzero subrings

$$B_t \subseteq B_{t-1} \subseteq \ldots \subseteq B_0 = B$$

that $B_t \in M_1^{(n+1)}$ and B_i is an ideal of the ring B_{i-1} for all $i = 1, 2, \ldots, t$ [5, p. 418, Lemma 1]. An ideal of a semiprime ring is itself a semiprime ring. Hence B_t is a semiprime ring. Thus there are only two possibilities: a) B_t is a finite commutative ring; b) the ring B_t is isomorphic to the ring A_n.

Let us consider the first case. It is clear that B_t is an artinian ring. Since B_t is a semiprime ring, it has an identity e. By Lemma 4, $Be = B_t e = B_t$. Hence B_t is an ideal of the ring B. Since $B_t \in M_1^{(n+1)} \subseteq M_n^{(n+1)}$, the statement (*) is proved.

Now let us consider the second case. We can assume that $B_t = A_n$. Let

$$P = B_t + BB_t + B_t B + BB_t B\,, \quad r(B; P) = \{b \in B / Pb = 0\}\,.$$

By Lemma 4, $P^m \subseteq B_t \subseteq P$ for some m. Hence we have

$$(**) \qquad K = r(B; P^m) \supseteq r(B; B_t) \supseteq r(B; P).$$

Further, $(PK)^m = PKPK \ldots PK \subseteq P^m K = 0$. Since B is a semiprime ring, $PK = 0$ and $K \subseteq r(B; P)$. Hence $K = r(B; B_t) = r(B; P)$ (see $(**)$). Therefore $r(B; B_t)$ is an ideal of the ring B. Let $L = \{b \in B / bK = 0\}$. It is clear that L is an ideal of the ring B and $L \cap K = 0$. Let $H_i = B_i \cap L$ for all $i = 1, 2, \ldots, t$. Evidently, $H_t = B_t$, H_i is an ideal of the ring H_{i-1} for all $1 \leqq i \leqq t$ and $H_0 = L$ is an ideal of the ring B. Further, $r(H_0; B_t) = K \cap L = 0$. Assume that $na = 0$ for some $0 \neq n \in Z$ and $0 \neq a \in H_0$. Since $B_t a \neq 0$ and $B_t an = 0$, $T = \{b \in P / bn = 0\} \neq 0$. It is clear that T is an ideal of the ring B and $T^m \subseteq P^m \subseteq B_t$. Since $xn \neq 0$ for all $0 \neq x \in B_t = A_n$, $T^m = 0$. But B is a semiprime ring. We obtain a contradiction. Therefore $na \neq 0$ for all $0 \neq a \in H_0$ and $0 \neq n \in Z$. Let $S = Z \setminus \{0\}$ and let $S^{-1}H_0$ be a localization of the ring H_0 (see [2]). By the above, $H_0 \subseteq S^{-1}H_0$ and

$$r(S^{-1}H_0; S^{-1}B_t) = S^{-1}\, r(H_0; B_t) = 0.$$

It is clear that $S^{-1}B_t = S^{-1}A_n = D$ and the identity of the field D will be the identity for all rings $S^{-1}H_i$, $i = 0, 1, \ldots, t$. Evidently, $S^{-1}H_i$ is an ideal of the ring $S^{-1}H_{i-1}$. Hence

$$D = S^{-1}H_t = S^{-1}H_{t-1} = \ldots = S^{-1}H_0, \quad H_0 \subseteq D.$$

Assume now that the ring H_0 does not contain identity. By Corollary 3, $H_0 = A_{n-r}$ for some $r \leqq t$. Since $A_0 \ni 1$, $n - r \neq 0$, $r < n$. Further, by Lemma 2, $A_{n-r} \in \in M_{r+1}^{(n+1)} \subseteq M_n^{(n+1)}$. Therefore in this case the condition $(*)$ holds.

Suppose now that the ring H_0 contains an identity e. By Corollary 3, the chain of subrings $A_n = H_t \subseteq H_{t-1} \subseteq \ldots \subseteq H_0$ has a form $A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_{n-r} \subseteq H_i = = H_{i-1} = \ldots = H_0$ and A_{n-r} is an ideal of the ring H_0 and $n - r \neq 0$. By Lemma 4, $eB = eH_0 = H_0$. Since $A_{n-r}e = A_{n-r}$, $A_{n-r}B = A_{n-r}eB = A_{n-r}H_0 \subseteq A_{n-r}$. Similarly $BA_{n-r} \subseteq A_{n-r}$. Hence A_{n-r} is an ideal of the ring B. It is clear that $A_{n-r} \subseteq \subseteq M_{r+1}^{(n+1)} \subseteq M_n^{(n+1)}$ (see Lemma 2). Therefore the condition $(*)$ holds in all cases. This completes our proof.

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Author's address: Department of Mathematics and Mechanics, Moscow State University, Moscow, 117 234, USSR.