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Czechoslovak Mathematical Journal, Vol. 32 (1982), No. 3, 481–487

Persistent URL: <http://dml.cz/dmlcz/101823>

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ON THE LIE ALGEBRA OF VERTICAL PROLONGATION OPERATORS

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(Received August 8, 1981)

We define a bracket of vertical prolongation operators on arbitrary fibered manifolds by means of a recently introduced operation of the so called strong difference, [2], [3], and we show that these operators constitute a Lie algebra. (Kosmann-Schwarzbach, [4], introduce such a bracket by means of the so called "linearization" of vertical differential section operators. However, one step of her construction cannot be applied to arbitrary fibered manifolds. Hence our construction is a purely geometrical treatment of the original idea by Y. Kosmann-Schwarzbach). Then we deduce some properties of a special class of vertical prolongation operators formed by generalized Lie derivatives of sections and morphisms. — All considerations are in the category C^∞ .

1. Strong difference. We shall need the concept of strong difference introduced in [2] or [3]. Given the second tangent bundle $T(TM) = TTM$ of an arbitrary manifold M , we have besides the bundle projection $p_{TM}: TTM \rightarrow TM$ also the tangent map $Tp_M: TTM \rightarrow TM$ to the bundle projection of the (first) tangent bundle $p_M: TM \rightarrow M$. In [2] and [3] the author has proved that every two vectors $A, B \in TTM$ satisfying $p_{TM}(A) = Tp_M(B)$ and $p_{TM}(B) = Tp_M(A)$ determine a vector in TM denoted by $A \dot{-} B$ and called the strong difference of A and B .

We apply the concept of the strong difference to the second vertical bundle VVY of an arbitrary fibered manifold Y over X . In this case we perform the construction described in [3] on each manifold Y_x (= the fiber over x). If $p: Y \rightarrow X$ is a fibered manifold and $q_Y: VY \rightarrow Y$ is its vertical bundle, we have besides the vector bundle $q_{VY}: VVY \rightarrow VY$ also a second vector bundle $Vq_Y: VVY \rightarrow VY$, where Vq_Y is the vertical tangent map to q_Y . According to [3], any two vectors $A, B \in VVY$ satisfy the conditions for the strong difference iff

$$(1) \quad Vq_Y(A) = q_{VY}(B), \quad Vq_Y(B) = q_{VY}(A).$$

In natural local coordinates $(x^i, y^p, Y^p, dy^p, dY^p)$ on VVY induced from some local coordinates (x^i, y^p) on Y , $A, B \in VVY$ satisfy the conditions for the strong difference iff $A \equiv (x^i, y^p, a^p, b^p, A^p)$, $B \equiv (x^i, y^p, b^p, a^p, B^p)$. Then $A \dot{-} B \equiv (x^i, y^p, A^p - B^p)$.

As a direct consequence of Theorem 1, [3], we obtain

Lemma 1. *Let $Y \rightarrow X$ and $Z \rightarrow X$ be two fibered manifolds over the same base and let $f: Y \rightarrow Z$ be a base-preserving morphism. If $A, B \in VVY$ satisfy the conditions for the strong difference, then $VVf(A), VVf(B) \in VVZ$ also satisfy the conditions for the strong difference, and*

$$VVf(A) \div VVf(B) = Vf(A \div B)$$

holds.

2. Lie algebra of vertical prolongation operators. Let $Y \rightarrow X$ and $Z \rightarrow X$ be two fibered manifolds over the same base. A *differential operator A of Y into Z* is a rule transforming each section s of Y into a section As of Z . Operator A is said to be of the order r , if the value $As(x)$ depends only on the r -jet $j_x^r s$, $x \in X$. In this case we obtain an associated base-preserving morphism $\mathcal{A}: J^r Y \rightarrow Z$ which is assumed to be smooth. By definition, we have $\mathcal{A}(j_x^r s) = As(x)$.

For any r -th order differential operator A of Y into Z , we define its vertical prolongation VA which is an r -th order differential operator of $VY \rightarrow X$ into $VZ \rightarrow X$. Any section $\sigma: X \rightarrow VY$ can be expressed, at least locally, as

$$\sigma = \left. \frac{\partial}{\partial t} \right|_0 s_t,$$

where s_t is a one-parameter family of sections of Y . Then we put

$$(2) \quad VA(\sigma) = VA \left(\left. \frac{\partial}{\partial t} \right|_0 s_t \right) = \left. \frac{\partial}{\partial t} \right|_0 As_t.$$

To demonstrate the global character of this definition, we use the associated morphism $\mathcal{A}: J^r Y \rightarrow Z$. The morphism $\mathcal{V}\mathcal{A}: J^r VY \rightarrow VZ$ associated with VA is of the form $\mathcal{V}\mathcal{A} = V\mathcal{A} \circ i_Y^r$, where $V\mathcal{A}: VJ^r Y \rightarrow VZ$ is the vertical tangent map to \mathcal{A} and $i_Y^r: J^r VY \rightarrow VJ^r Y$ is the canonical identification defined by

$$i_Y^r \left(j_x^r \left. \frac{\partial}{\partial t} \right|_0 s_t \right) = \left. \frac{\partial}{\partial t} \right|_0 j_x^r s_t$$

for every one-parameter family of sections of Y and every $x \in X$, see [1]. Indeed,

$$\begin{aligned} VA(\sigma(x)) &= VA \left(\left. \frac{\partial}{\partial t} \right|_0 s_t(x) \right) = \left. \frac{\partial}{\partial t} \right|_0 As_t(x) = \left. \frac{\partial}{\partial t} \right|_0 (\mathcal{A}(j_x^r s_t)) = V\mathcal{A} \left(\left. \frac{\partial}{\partial t} \right|_0 j_x^r s_t \right) = \\ &= V\mathcal{A} \circ i_Y^r \left(j_x^r \left. \frac{\partial}{\partial t} \right|_0 s_t \right). \end{aligned}$$

If Z is a fibered manifold $q: Z \rightarrow Y$ over Y and A satisfies $qAs = s$ for all sections of Y , then A will be called a *prolongation differential operator*. In [4], prolongation differential operators are called differential section operators. A prolongation

differential operator A is characterized by the property that $\mathcal{A}: J^r Y \rightarrow Z$ is a morphism over id_Y . Then vertical prolongation VA is a prolongation differential operator with respect to projection $Vq: VZ \rightarrow VY$. Indeed,

$$Vq(VA(\sigma)) = Vq\left(VA\left(\frac{\partial}{\partial t}\Big|_0 s_t\right)\right) = Vq\left(\frac{\partial}{\partial t}\Big|_0 As_t\right) = \frac{\partial}{\partial t}\Big|_0 (qAs_t) = \frac{\partial}{\partial t}\Big|_0 s_t = \sigma$$

for every section $\sigma: X \rightarrow VY$.

Consider now the case $Z = VY$. A prolongation differential operator of Y into VY will be called a *vertical prolongation operator on Y* . Such operators form a real vector space, provided one defines

$$(k_1 A + k_2 B)s := k_1 As + k_2 Bs, \quad k_1, k_2 \in \mathbb{R}$$

for every section s of Y . Each vertical prolongation operator A on Y can be prolonged in the above sense into an operator VA of VY into VVY . If A, B are two vertical prolongation operators on Y , we can construct $VA(Bs), VB(As): X \rightarrow VVY$ for every section s of Y . Both $VA \circ B$ and $VB \circ A$ are prolongation differential operators of Y into VVY . We have $Vq_Y(VA(Bs)) = Bs$ since VA is a prolongation differential operator with respect to the projection Vq_Y . Further, if Bs is tangent to $s_t, s_0 = s$, then

$$q_{VY}(VA(Bs)) = q_{VY}\left(\frac{\partial}{\partial t}\Big|_0 As_t\right) = As.$$

Similarly we have $Vq_Y(VB(As)) = As, q_{VY}(VB(As)) = Bs$. Hence the conditions (1) for the strong difference are satisfied. If A and B are vertical prolongation operators of orders r and s , then $VA \circ B$ and $VB \circ A$ are prolongation differential operators of order $r + s$. The associated morphism to $VA \circ B$ is $VA \circ i_Y^r \circ J^r \mathcal{B}: J^{r+s} Y \rightarrow VVY$ because of $VA(Bs(x)) = V\mathcal{A} \circ i_Y^r \circ J^r(\mathcal{B} \circ J_x^s s) = V\mathcal{A} \circ i_Y^r \circ J^r \mathcal{B}(J_x^{r+s} s)$, where $J^r \mathcal{B}$ is the restriction of the r -th jet prolongation of \mathcal{B} to $J^{r+s} Y \subset J^r(J^s Y)$. Similarly, the associated morphism to $VB \circ A$ is $V\mathcal{B} \circ i_Y^s \circ J^s \mathcal{A}: J^{r+s} Y \rightarrow VVY$.

We now define a vertical prolongation operator of order $r + s$ on Y , called the bracket of A, B , by

$$(3) \quad [A, B](s) := VA(Bs) \dot{-} VB(As): X \rightarrow VVY.$$

The associated morphism to $[A, B]$ is $[\mathcal{A}, \mathcal{B}] = V\mathcal{A} \circ i_Y^r \circ J^r \mathcal{B} \dot{-} V\mathcal{B} \circ i_Y^s \circ J^s \mathcal{A}: J^{r+s} Y \rightarrow VVY$.

We are going to define the “value” of a vertical prolongation operator on any function $f: Y \rightarrow R$. Denote by $\delta f: VY \rightarrow R$ the fiber differential of f , [1]. For a vertical prolongation operator A with the associated morphism \mathcal{A} we put

$$(4) \quad Af := \delta f \circ \mathcal{A}: J^r Y \rightarrow R.$$

Lemma 2. *If A, B are two vertical prolongation operators of order r such that $Af = Bf$ holds for every function $f: Y \rightarrow R$, then $A \equiv B$.*

Proof. In local coordinates (x^i, y^p) on Y and the induced coordinates (x^i, y^p, Y^p) on VY and $(x^i, y^p, y_i^p, \dots, y_{i_1 \dots i_r}^p)$ on $J^r Y$, we have $f \equiv f(x^i, y^p)$, $\mathcal{A} \equiv x^i = x^i$, $y^p = y^p$, $Y^p = A^p(x^i, y^p, \dots, y_{i_1 \dots i_r}^p)$ and a similar expression for \mathcal{B} . Then

$$Af = \frac{\partial f}{\partial y^q} A^q, \quad Bf = \frac{\partial f}{\partial y^q} B^q.$$

Setting $f = y^p$, we obtain $A^p = B^p$ for every p , which implies $\mathcal{A} \equiv \mathcal{B}$. QED.

Further, if $f: J^s Y \rightarrow R$ is a function, we have the fiber differential $\delta f: VJ^s Y \rightarrow R$. Taking into account $J^s \mathcal{A}: J^{r+s} Y \rightarrow J^s VY$, we define

$$(5) \quad Af := \delta f \circ i_Y^s \circ J^s \mathcal{A}: J^{r+s} Y \rightarrow R.$$

Using (4) and (5), we shall prove

Proposition 1. *The set of all vertical prolongation operators on an arbitrary fibered manifold forms a Lie algebra with respect to the bracket defined by (3).*

Proof. Any function $f: Y \rightarrow R$ can be considered as a base-preserving morphism $f: Y \rightarrow X \times R$, where $X \times R \rightarrow X$ is the product fibered manifold. Then δf is the second component of the vertical tangent map $Vf: VY \rightarrow V(X \times R) = X \times TR$ and Af is the second component of $Vf \circ \mathcal{A}: J^r Y \rightarrow X \times TR$. Further, according to (5), $B(Af)$ is the fourth component of

$$VVf \circ V\mathcal{A} \circ i_Y^r \circ J^r \mathcal{B}: J^{r+s} Y \rightarrow X \times TTR.$$

Similarly, $A(Bf)$ is the fourth component of

$$VVf \circ V\mathcal{B} \circ i_Y^s \circ J^s \mathcal{A}: J^{r+s} Y \rightarrow X \times TTR.$$

But $V\mathcal{A} \circ i_Y^r \circ J^r \mathcal{B}$ and $V\mathcal{B} \circ i_Y^s \circ J^s \mathcal{A}$ satisfy the conditions for the strong difference and Lemma 1 implies

$$\begin{aligned} & VVf \circ V\mathcal{A} \circ i_Y^r \circ J^r \mathcal{B} - VVf \circ V\mathcal{B} \circ i_Y^s \circ J^s \mathcal{A} = \\ & = Vf(V\mathcal{A} \circ i_Y^r \circ J^r \mathcal{B} - V\mathcal{B} \circ i_Y^s \circ J^s \mathcal{A}) = Vf \circ [A, B]. \end{aligned}$$

Hence the second component of the latter map is $[A, B]f$. On the other hand, the second component of $VVf \circ V\mathcal{A} \circ i_Y^r \circ J^r \mathcal{B} - VVf \circ V\mathcal{B} \circ i_Y^s \circ J^s \mathcal{A}$ is $B(Af) - A(Bf)$, so that

$$(6) \quad [A, B]f = B(Af) - A(Bf).$$

Using (6), one finds easily

$$([[A, B], C] + [[B, C], A] + [[C, A], B])f = 0.$$

By Lemma 2, we deduce the Jacobi identity. QED.

3. Generalized Lie derivatives of sections and morphisms. Let $Y \rightarrow X$ be a fibered

manifold, η a projectable vector field on Y over a vector field ξ on X and s a section of Y . Then the Lie derivative (see [2]) of s with respect to η is

$$\mathcal{L}_\eta s = Ts \circ \xi - \eta \circ s : X \rightarrow VY.$$

Thus, every projectable vector field η transforms any section s of Y into a section $\mathcal{L}_\eta s : X \rightarrow VY$ and we can consider \mathcal{L}_η as a vertical prolongation operator of order 1.

According to Proposition 1 the set of the Lie derivatives of sections with respect to the projectable vector fields forms a Lie algebra. The bracket is defined by the following formula

$$(7) \quad [\mathcal{L}_\eta, \mathcal{L}_{\bar{\eta}}] s = V\mathcal{L}_\eta \mathcal{L}_{\bar{\eta}} s - V\mathcal{L}_{\bar{\eta}} \mathcal{L}_\eta s.$$

Lemma 3. For every projectable vector field η on Y and every section $\Phi : X \rightarrow VY$ we have

$$(8) \quad (V\mathcal{L}_\eta) \Phi = i \circ \mathcal{L}_{V_\eta} \Phi,$$

where i is the canonical involution of VY and vector field V_η is the vertical prolongation of η defined by means of the vertical prolongation of the flow of η .

Proof. In local coordinates, $\Phi \equiv (x^i, \varphi^p(x), \Phi^p(x))$. Then $V\mathcal{L}_\eta$ transforms Φ into

$$\left(x^i, \varphi^p, (\mathcal{L}_\eta \varphi)^p, \Phi^p, \frac{\partial \Phi^p}{\partial x^i} \xi^i - \frac{\partial \eta^p}{\partial y^q} \Phi^q \right).$$

Further,

$$V_\eta \equiv \xi^i(x) \frac{\partial}{\partial x^i} + \eta^p(x, y) \frac{\partial}{\partial y^p} + \frac{\partial \eta^p}{\partial y^q} Y^q \frac{\partial}{\partial Y^p}$$

and

$$\mathcal{L}_{V_\eta} \Phi \equiv \left(\frac{\partial \varphi^p}{\partial x^i} \xi^i - \eta^p \right) \frac{\partial}{\partial y^p} + \left(\frac{\partial \Phi^p}{\partial x^i} \xi^i - \frac{\partial \eta^p}{\partial y^q} \Phi^q \right) \frac{\partial}{\partial Y^p}.$$

Thus \mathcal{L}_{V_η} transforms Φ into

$$\left(x^i, \varphi^p, \Phi^p, (\mathcal{L}_\eta \varphi)^p, \frac{\partial \Phi^p}{\partial x^i} \xi^i - \frac{\partial \eta^p}{\partial y^q} \Phi^q \right). \quad \text{QED.}$$

Owing to Lemma 3, the bracket of Lie derivatives of sections can be expressed equivalently as

$$(9) \quad [\mathcal{L}_\eta, \mathcal{L}_{\bar{\eta}}] (s) = \mathcal{L}_{V_\eta} \mathcal{L}_{\bar{\eta}} s - \mathcal{L}_{V_{\bar{\eta}}} \mathcal{L}_\eta s.$$

In [2], Kolář has proved

$$(10) \quad \mathcal{L}_{V_\eta} \mathcal{L}_{\bar{\eta}} s - \mathcal{L}_{V_{\bar{\eta}}} \mathcal{L}_\eta s = \mathcal{L}_{[\eta, \bar{\eta}]} s.$$

From (9) and (10) it follows that $\mathcal{L}_{[\eta, \bar{\eta}]} s = [\mathcal{L}_\eta, \mathcal{L}_{\bar{\eta}}] (s)$ and we can consider \mathcal{L} as a

homomorphism of the Lie algebra of all projectable vector fields on Y into the Lie algebra of all vertical prolongation operators on Y .

Let $Y \rightarrow X$ and $Z \rightarrow X$ be two fibered manifolds over the same base. Let η be a projectable vector field on Y over a vector field ξ on X and ζ a projectable vector field on Z over the same vector field ξ . Then for every base-preserving morphism $f: Y \rightarrow Z$ we define its Lie derivative with respect to η and ζ by

$$\mathcal{L}_{(\eta, \zeta)} f := Tf \circ \eta - \zeta \circ f : Y \rightarrow VZ.$$

Remark. Let $G \rightarrow^q Z \rightarrow^p X$ be a double fibered manifold and $f: Y \rightarrow Z$ a base-preserving morphism. A *prolongation differential f -operator* of Y into G is a rule transforming each section s of Y into a section $As: X \rightarrow G$ such that $q(As) = f \circ s$. (Kosmann-Schwarzbach, [4], calls such an operator a differential section f -operator.) In the special case of $VZ \rightarrow^q Z \rightarrow^p X$, a prolongation differential f -operator will be called a vertical prolongation f -operator. Lie derivatives of morphisms can be considered as vertical prolongation f -operators transforming each section s of Y into a section $(\mathcal{L}_{(\eta, \zeta)} f) \circ s : X \rightarrow VZ$.

Let $\bar{\eta}$ and $\bar{\xi}$ be another pair of projectable vector fields on Y and Z over the same vector field $\bar{\xi}$ on X . Then we define the iterated Lie derivative by

$$(11) \quad \mathcal{L}_{(\bar{\eta}, V\bar{\xi})} \mathcal{L}_{(\eta, \zeta)} f : Y \rightarrow VVZ.$$

Proposition 2. $\mathcal{L}_{(\eta, V\zeta)} \mathcal{L}_{(\bar{\eta}, \bar{\xi})} f$ and $\mathcal{L}_{(\bar{\eta}, V\bar{\xi})} \mathcal{L}_{(\eta, \zeta)} f$ satisfy the conditions for the strong difference and

$$(12) \quad \mathcal{L}_{(\eta, V\zeta)} \mathcal{L}_{(\bar{\eta}, \bar{\xi})} f \dot{-} \mathcal{L}_{(\bar{\eta}, V\bar{\xi})} \mathcal{L}_{(\eta, \zeta)} f = \mathcal{L}_{((\eta, \bar{\eta}), (\zeta, \bar{\xi}))} f.$$

Proof. The generalized Lie derivative of a map $f: M \rightarrow N$ with respect to a pair of vector fields ξ and η on M and N is defined by the formula $\mathcal{L}_{(\xi, \eta)} f := Tf \circ \xi - \eta \circ f : M \rightarrow TN$, [3]. In general, it is easy to see that if the values of f lie in a submanifold $Q \subset N$ and the vector field η is tangent to Q , then $\mathcal{L}_{(\xi, \eta)} f = \mathcal{L}_{(\xi, \bar{\eta})} f$, where $\bar{\eta}$ is the restriction of η to Q . Since ζ is a projectable vector field on Z and the values of $\mathcal{L}_{(\bar{\eta}, \bar{\xi})} f$ lie in VZ , we have, in our case, $\mathcal{L}_{(\eta, V\zeta)} \mathcal{L}_{(\bar{\eta}, \bar{\xi})} f = \mathcal{L}_{(\eta, V\zeta)} \mathcal{L}_{(\bar{\eta}, \bar{\xi})} f$, where $T\bar{\xi}$ is the prolongation of $\bar{\xi}$ with respect to the tangent functor constructed by means of flows, [3]. Our proposition is then a special case of Theorem 2 of [3]. QED.

Owing to Proposition 2, the set of Lie derivatives of morphisms forms the Lie algebra with a bracket defined by

$$(13) \quad [\mathcal{L}_{(\eta, \zeta)}, \mathcal{L}_{(\bar{\eta}, \bar{\xi})}] f = \mathcal{L}_{(\eta, V\zeta)} \mathcal{L}_{(\bar{\eta}, \bar{\xi})} f \dot{-} \mathcal{L}_{(\bar{\eta}, V\bar{\xi})} \mathcal{L}_{(\eta, \zeta)} f.$$

A simpler situation occurs, if $Z = E$ is a vector bundle. Then $\mathcal{L}_{(\eta, \zeta)} f$ can be considered as a base-preserving morphism of Y into E as well. Hence we can construct the iterated Lie derivative $\mathcal{L}_{(\eta, \zeta)} \mathcal{L}_{(\bar{\eta}, \bar{\xi})} f : Y \rightarrow E$. A projectable vector field ζ on E is called linear, [2], if its flow is formed by linear fiber isomorphisms.

Proposition 3. *Let $Y \rightarrow X$ be a fibered manifold, $E \rightarrow X$ a vector bundle over the same base, and $f: Y \rightarrow E$ a base-preserving morphism. Let $\eta, \bar{\eta}$ be two projectable vector fields on Y over vector fields $\xi, \bar{\xi}$ on X , and $\zeta, \bar{\zeta}$ two linear vector fields on E over the same vector fields $\xi, \bar{\xi}$, respectively. Then*

$$(14) \quad \mathcal{L}_{(\bar{\eta}, \bar{\zeta})} \mathcal{L}_{(\eta, \zeta)} f - \mathcal{L}_{(\eta, \zeta)} \mathcal{L}_{(\bar{\eta}, \bar{\zeta})} f = \mathcal{L}_{([\bar{\eta}, \eta], [\bar{\zeta}, \zeta])} f.$$

Proof represents an easy direct calculation in local coordinates. We remark that in the vector bundle case we do not need the strong difference. However, the condition of linearity of ζ and $\bar{\zeta}$ is essential, i.e. (14) does not hold for general projectable vector fields ζ and $\bar{\zeta}$.

References

- [1] *H. Goldschmidt, S. Sternberg*: The Hamilton-Cartan formalism in the calculus of variations, *Ann. Inst. Fourier (Grenoble)*, 23 (1973), 203–267.
- [2] *I. Kolář*: Fundamental vector fields on associated fiber bundles, *Čas. pro pěst. mat.*, 102 (1977), 411–425.
- [3] *I. Kolář*: On the second tangent bundle and generalized Lie derivatives, to appear in *Tensor*.
- [4] *Y. Kosmann-Schwarzbach*: Vector Fields and Generalized Vector Fields on Fibered Manifolds, in: *Geometry and Differential Geometry, Proceedings, Lecture Notes in Mathematics 792* (1980), 307–355.

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