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SIMULTANEOUS SOLUTIONS OF A SYSTEM OF ABEL EQUATIONS AND DIFFERENTIAL EQUATIONS WITH SEVERAL DEVIATIONS

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I. In this paper we give necessary and sufficient conditions under which a transformation of the independent variable exists that changes a differential equation (single or system, linear or nonlinear) with several deviating arguments, f_i, $i = 1, \ldots, k$, into a differential equation with constant deviations.

The problem leads to finding a simultaneous solution of a system of Abel equation (Problem 195, [6, p. 308]), and the method is based on 0. Borůvka's result concerning one-parameter continuous groups of transformations on line [1].

Let $A_n(f_1, \ldots, f_k)$ denote a differential equation of the n-th order with k deviating arguments $f_i(x)$, $f_i : I \to^{\text{onto}} I = (a, b)$ an open real interval (the cases $a = -\infty$ and $b = \infty$ are not excluded), let $f_i \in C^n(I)$, $n \geqq 1$, $df_i(x)/dx > 0$ on I, $i = 1, \ldots, k$. Here $C^m(S)$ is the set of all continuous functions on S continuously differentiable up to and including the m-th order, $m = 0$ means continuity.

Consider a transformation of the equation $A_n(f_1, \ldots, f_k)$ into $B_n(h_1, \ldots, h_k)$ consisting in a change $x \mapsto \phi(x) = t$ of the independent variable, ϕ being a bijection of the interval I onto J, $\phi \in C^n(I)$, $d\phi(x)/dx > 0$ on I. Hence, if $y : x \mapsto y(x)$ denotes a solution of $A_n(f_1, \ldots, f_k)$ on I, then $z = y\phi^{-1} : J \to \mathbf{R}$ is a solution of $B_n(h_1, \ldots, h_k)$ on J.

In accordance with M. Kuczma [4, p. 13], we use the following notation: upper indices at the sign of a function denote iterations,

i.e.
$$f^1(x) = f(x)\,, \quad f^0(x) = x\,, \quad f^{n+1}(x) = f(f^n(x))\,,$$
$$f^{n-1}(x) = f^{-1}(f^n(x))\,, \quad f^{-1} \text{ denoting the inverse to } f\,;$$
$$f', f'', f''', \ldots, f^{(n)} \quad \text{are the derivatives of } f\,.$$

In [5] it was shown that $y^{(j)}(f_i(x))$ can always be expressed in terms of $\{z^{(s)}(h_i(t)), s \leqq j\}$, where h_i satisfies
$$\phi f_i(x) = h_i\, \phi(x)\,, \quad x \in I\,.$$
Moreover, if $A_n(f_1, \ldots, f_k)$ is linear, then $B_n(h_1, \ldots, h_k)$ is also linear.

The existence of a transformation ϕ that converts a given equation $A_n(f_1, \ldots, f_k)$ into an equation $B_n(h_1, \ldots, h_k)$ with constant deviations, i.e.,

$$h_i(t) = t + c_i, \ c_i \text{ being constants},$$

is therefore equivalent to the existence of a common solution ϕ of the following system of functional equations

$$\phi f_i(x) = \phi(x) + c_i, \quad i = 1, \ldots, k, \quad x \in I. \tag{1}$$

With respect to (1) we are looking for conditions under which k functions f_i can be embedded into a one-parameter group of transformations $\phi^{-1}(\phi(x) + c)$, $c \in \mathbf{R}$.

II. Throughout this section we will suppose that the system (1) has a solution $\phi : I \to \mathbf{R}$, $\phi \in C^n(I)$, $n \geqq 1$, $\mathrm{d}\phi(x)/\mathrm{d}x > 0$ on I.

Proposition 1. *All functions f_i and f_j commute, i.e.,*

$$f_i f_j(x) = f_j f_i(x) \quad \textit{on } I \textit{ for all pairs } (i, j).$$

Proof. Under our supposition,

$$f_i(x) = \phi^{-1}(\phi(x) + c_i) \quad \text{and} \quad f_j(x) = \phi^{-1}(\phi(x) + c_j).$$

Since $f_i(I) = I$ and $f_j(I) = I$, both $f_i f_j$ and $f_j f_i$ are defined and

$$f_i f_j(x) = \phi^{-1}(\phi(x) + c_i + c_j) = f_j f_i(x)$$

holds on I for each i and j, $1 \leqq i, j \leqq k$, q.e.d.

Denote by F the set of all finite compositions of f_i and their inverses f_i^{-1}, $i = 1, \ldots, k$. The following corollary is a direct consequence of Proposition 1.

Corollary. *The set F can be expressed as*

$$F = \{f_1^{s_1} f_2^{s_2} \ldots f_k^{s_k};\ s_i \textit{ integers}\}, \tag{2}$$

and any two $g_1, g_2 \in F$ commute.

Proposition 2. *Let $g_1, g_2 \in F$. If there exists an $x_0 \in I$, such that*

$$g_1(x_0) = g_2(x_0),$$

then

$$g_1(x) = g_2(x) \quad \textit{for all} \quad x \in I.$$

Proof. For $g_1, g_2 \in F$ we have

$$g_1(x) = f_1^{s_1} f_2^{s_2} \ldots f_k^{s_k}(x) = \phi^{-1}(\phi(x) + \sum_{i=1}^{k} s_i c_i)$$

and

$$g_2(x) = f_1^{r_1} f_2^{r_2} \ldots f_k^{r_k}(x) = \phi^{-1}(\phi(x) + \sum_{i=1}^{k} r_i c_i),$$

$s_1, \ldots, s_k$, $r_1, \ldots, r_k$ being integers. If $g_1(x_0) = g_2(x_0)$, then $\sum_{i=1}^{k} s_i c_i = \sum_{i=1}^{k} r_i c_i$, or $g_1(x) = g_2(x)$ for all $x \in I$, q.e.d.

Define the set D as the union of graphs of all functions $g \in F$, i.e.,

$$D := \{(x, y);\ \text{there exists } g \in F \text{ such that } g(x) = y\}\,. \tag{3}$$

Evidently $D \subset I^2$.

Proposition 3. *The set D is dense in I^2 if and only if there exists at least one pair (i, j), $1 \leqq i, j \leqq k$, such that the quotient c_i/c_j is irrational.*

Proof. If all $c_i = 0$, then $f_i = \mathrm{id}$ on I for all i, and $D = \{(x, x);\ x \in I\}$ is not dense in I^2.

Let $c_i \neq 0$ for an i, $1 \leqq i \leqq k$. Since f_i^m is defined for all integers m, $\phi f_i^m(x_0) = \phi(x_0) + mc_i$ for $x_0 \in \mathbf{R}$. Then $\phi(I) = \mathbf{R}$, because ϕ is continuous and

$$\lim_{m \to \pm\infty} \phi(f_i^m(x_0)) = \pm\infty\,.$$

The transformation $T: (x, y) \mapsto (\phi(x), \phi(y))$, $(x, y) \in I^2$, is a diffeomorphism of I^2 onto $\mathbf{R}^2$, because $\mathrm{d}\phi(x)/\mathrm{d}x > 0$ on I. Moreover, $T(D) = \{(t, t + \sum_{i=1}^{k} m_i c_i);\ t \in \mathbf{R},\ m_i \in \mathbf{Z}\}$, and it is dense in $\mathbf{R}^2$ if and only if at least one quotient c_i/c_j is irrational. Since T is a diffeomorphism, $T(D)$ is dense in $\mathbf{R}^2$ exactly when D is dense in I^2, q.e.d.

Proposition 4. *If D is not dense in I^2, then there exists a $\mu \in C^n(I)$ such that $\mathrm{d}\mu(x)/\mathrm{d}x > 0$, $\mu(x) > x$ on I and*

$$f_i = \mu^{m_i}$$

holds for each i, $1 \leqq i \leqq k$, and suitable $m_i \in \mathbf{Z}$.

Proof. If D is not dense in I^2, then, due to Proposition 3, all quotients c_i/c_j $(c_j \neq 0)$ are rational. Hence there exists $d > 0$, such that $c_i = m_i d$ for all i and for suitable integers m_i. If μ is defined by

$$\mu(x) := \phi^{-1}(\phi(x) + d)\,, \quad x \in I\,,$$

then $\mu \in C^n(I)$, $\mathrm{d}\mu/\mathrm{d}x > 0$, $\mu(x) > x$, and $f_i(x) = \phi^{-1}(\phi(x) + c_i) = \phi^{-1}(\phi(x) + m_i d) = \mu^{m_i}(x)$, q.e.d.

Define the function $H: D \to \mathbf{R}$ by

$$H(x, y) = g'(x)\,, \quad \text{where } g \in F \text{ and } g(x) = y\,. \tag{4}$$

Proposition 5. *The function H is well defined by (4) and it satisfies*

$$H(x, y) > 0 \quad \textit{for all} \quad (x, y) \in D\,, \quad \textit{and} \tag{5}$$
$$H(x, y)\, H(y, z) = H(x, z) \quad \textit{if} \quad (x, y) \cup (y, z) \subset D\,.$$

Proof. By Proposition 2, for each $(x, y) \in D$ there exists just one function $g \in F$ satisfying $g(x) = y$ (even if g can be written in different ways as a composition of f_i's). Hence H is well defined. The positivity of H follows from (2) and from $\mathrm{d}f_i(x)\,\mathrm{d}x > 0$ on I for all i.

Finally, if $(x, y) \cup (y, z) \subset D$, then there exist $g_1, g_2 \in F$ such that $g_1(x) = y$, $g_2(y) = z$. Then $g_2\, g_1(x) = z$, $g_2\, g_1 \in F$, and $(x, z) \in D$. Since

$$(g_2 g_1)' = g_2'(g_1) \cdot g_1' ,$$

we have $H(x, z) = H(x, y) \,.\, H(y, z)$, q.e.d.

The definition of H yields the following property.

Proposition 6. *Each $g \in F$ (in particular, each f_i) is a solution of the differential equation*

$$y' = H(x, y) , \quad (x, y) \in D .$$

The next property is a direct consequence of O. Borůvka's result [2].

Proposition 7. *There exists an extension H^* of H to I^2 ($H = H^*$ on D) such that $H^* \in C^{n-1}(I^2)$, and*

(5*) $\quad H^*(x, y) > 0$ *on* I^2 *and* $H^*(x, y) \,.\, H^*(y, z) = H^*(x, z)$ *on* I^2

holds.

Proof. We suppose the existence of a solution ϕ of (1) satisfying $\phi \in C^n(I)$, $\mathrm{d}\phi(x)/\mathrm{d}x > 0$ on I. In accordance with [2], define

$$H^*(x, y) = \phi'(x)/\phi'(y) \quad \text{on} \quad I^2 .$$

Evidently $H^* \in C^{n-1}(I^2)$, and (5*) hold. For $(x, y) \in D$, there exists $g \in F$ such that $g(x) = y$. In view of (2), we have

$$\phi(g(x)) = \phi(x) + \text{const.}$$

Then $H^*(x, y) = \phi'(x)/\phi'(y) = \phi'(x)/\phi'(g(x)) = g'(x) = H(x, y)$ on D, q.e.d.

Remark. If D is dense on I^2, then H^* is uniquely determined by H, because H^* is continuous.

III. We may summarize the results of Section II in the following way.

Theorem 1. *Let $I = (a, b)$ be an open interval of reals, $f_i : I \to^{\text{onto}} I$, $f_i \in C^n(I)$ for some $n \geqq 1$, and $\mathrm{d}f_i/\mathrm{d}x > 0$ on I, $i = 1, \ldots, k$. Suppose that the system (1) of Abel functional equations has a solution ϕ, $\phi \in C^n(I)$, $\mathrm{d}\phi(x)/\mathrm{d}x > 0$ on I.*

Then the sets F, D and the function $H : D \to \mathbf{R}$ are well defined by (2), (3), and (4).

If D is not dense in I^2, then there exists a function $\mu \in C^n(I)$, $\mathrm{d}\mu(x)/\mathrm{d}x > 0$, $\mu(x) > x$ on I such that

$$f_i = \mu^{m_i}$$

for each i and suitable $m_i \in \mathbf{Z}$.

If D is dense in I^2, then H can be uniquely extended to a continuous H^* on I^2. This H^* is in $C^{n-1}(I^2)$ and satisfies (5).*

Now, we shall prove the following

Theorem 2. *If $f_i = \mu^{m_i}$ for $i = 1, \ldots, k$, where m_i are integers, and $\mu : I \to^{\text{onto}} I = (a, b) \subset \mathbf{R}$, $\mu \in C^n(I)$, $\mathrm{d}\mu(x)/\mathrm{d}x > 0$ on I, then there exists a solution ϕ of (1), $\phi \in C^n(I)$, $\mathrm{d}\phi(x)/\mathrm{d}x > 0$ on I.*

Proof. In [3] it was shown that under our assumptions on μ, there exists a solution ϕ of

$$\phi\, \mu(x) = \phi(x) + \operatorname{sign}(\mu(x) - x)$$

satisfying $\phi \in C^n(I)$, $\mathrm{d}\phi(x)/\mathrm{d}x > 0$ on I (even solutions depending on an arbitrary function). At the same time this ϕ is a solution of the system (1), since $\phi\, f_i(x) = \phi\, \mu^{m_i}(x) = \phi(x) + m_i \operatorname{sign}(\mu(x) - x)$, $i = 1, \ldots, k$, q.e.d.

For the case when D is dense in I^2 and $H^* \in C^{n-1}(I^2)$, we may utilize O.Borůvka's result [2]. For the sake of completeness we recall it here.

If $H^* \in C^{n-1}(I^2)$ and (5*) is satisfied, define

$$\phi(x) := k \int_{x_0}^{x} H^*(\sigma, y_0)\, \mathrm{d}\sigma\,, \quad x_0, y_0, x \in I\,, \quad k > 0\,.$$

Then

$$\phi \in C^n(I)\,, \quad \mathrm{d}\phi(x)/\mathrm{d}x > 0\,,$$

and

$$\begin{aligned}\mathrm{d}\phi f_i\, \phi^{-1}(t) &= kH^*(f_i\, \phi^{-1}(t), y_0) \cdot f_i'(\phi^{-1}(t)) \cdot [kH^*(\phi^{-1}(t), y_0)]^{-1} = \\ &= H^*(f_i\, \phi^{-1}(t), y_0) \cdot H(\phi^{-1}(t), f_i(\phi^{-1}(t))) \cdot H^*(y_0, \phi^{-1}(t)) = 1\,,\end{aligned}$$

or $\phi f_i\, \phi^{-1}(t) = t + \text{const}$. Hence ϕ is the required solution of the system (1).

IV. Example 1. Consider the differential equation $y'(x) = y(\sqrt{x}) + y(x^4)$, $x \in (1, \infty)$, with two deviating arguments, $f_1(x) = x^{1/2}$, $f_2(x) = x^4$. The set F of all finite compositions of f_1, f_2, f_1^{-1}, and f_2^{-1} is

$$F = \{x^{2^s};\ s \text{ an integer},\ x \in (1, \infty)\}\,.$$

The set $D = \{(x, x^{2^s});\ s \in \mathbf{Z},\ x \in (1, \infty)\}$ is not dense in $(1, \infty)^2$. Any $g \in F$ satisfies $g = f_1^m$ for a suitable $m \in \mathbf{Z}$. Hence any solution ϕ, $\phi \in C^n(1, \infty)$, $\mathrm{d}\phi(x)/\mathrm{d}x > 0$ on $(1, \infty)$ of

$$\phi(\sqrt{x}) = \phi(x) + 1$$

transforms our differential equation into an equation with constant deviations, since $\phi(x^4) = \phi(x) - 2$.

Example 2. Consider $y'(x) = y(\sqrt{x}) + y(x^3)$, $x \in (1, \infty)$, with $f_1(x) = x$, $f_2(x) = x^3$. The set F is given by

$$F = \{x^{2^s x^{3^r}};\ x \in (1, \infty),\ r, s \in \mathbf{Z}\}\,,$$

and

$$D = \{(x, x^\alpha);\ \alpha = 2^s 3^r,\ s, r \in \mathbf{Z}\} \quad \text{is dense in} \quad (1, \infty)^2 .$$

We have

$$H(x, y) = H(x, x^\alpha) = \mathrm{d}x^\alpha/\mathrm{d}x = \alpha x^{\alpha - 1} \quad \text{on} \quad D .$$

Hence

$$H^*(x, y) = H^*(x, x^\beta) = \beta x^{\beta - 1} \quad \text{on} \quad I^2 \quad \text{for all} \quad \beta \in \mathbf{R}^+ ,$$

$$H^*(x, y) = \frac{\ln y}{\ln x}\, x^{((\ln y/\ln x) - 1)} = \frac{y \ln y}{x \ln x} \quad \text{on} \quad (1, \infty)^2 .$$

Now $\phi(x) = k \int_{x_0}^{x} H^*(\sigma, y_0)\,\mathrm{d}\sigma = K_1 \ln \ln x + K_2$, $K_1 > 0$, and

$$\phi(\sqrt{x}) = \phi(x) - K_1 \ln 2 ,$$

$$\phi(x^3) = \phi(x) + K_1 \ln 3 .$$

There are no integers r and s such that $2^r = 3^s$.

The solution ϕ transforms our differential equation with deviations $\sqrt{x}$ and x^3 into an equation with constant deviations $t - K_1 \ln 2$ and $t + K_1 \ln 3$.

Let us note that this solution ϕ is also one of the solutions in the preceding example.

V. We have seen that, when $f_i \in C^n(I)$, $f_i'(x) > 0$ on I, $f_i(I) = I$, $f_i f_j = f_j f_i$, and

$$f_1^{s_1} \ldots f_k^{s_k}(x_0) = x_0 \quad \text{implies} \quad f_1^{s_1} \ldots f_k^{s_k}(x) = x \quad \text{for all} \quad x \in I ,$$

then H on D is well defined and satisfies (5). However, the following questions are open. If F is not a group with one generator (as in Proposition 4), is it always possible to extend H to a continuous H^* on I^2 without supposing the existence of a solution ϕ of (1)? If such a continuous extension H^* exists, then it is unique. Of what class is this extension?

Let us make the following remarks to the problem.

If H^* is at least from $C^0(I^2)$, each $g \in F$ can be written as

$$g(x) = \phi^{-1}(\phi(x) + \alpha) \in C^n(I) , \quad \text{for suitable} \quad \alpha \in \mathbf{R} ,$$

where

$$\phi(x) = k \int_{x_0}^{x} H^*(\sigma, y_0)\,\mathrm{d}\sigma .$$

Define $h_\alpha(x) := \phi^{-1}(\phi(x) + \alpha)$, for all $\alpha \in \mathbf{R}$. Evidently $\{(x \mapsto h_\alpha(x));\ \alpha \in \mathbf{R}\} \supset F$. If the system h_α is considered as depending on α, $\alpha \to h_\alpha(x)$ is of the class $C^1(\mathbf{R})$ only. By introducing a new parametrization of α, $\alpha = \phi(\beta)$, we may improve the smoothness of the dependence on a parameter even to the class C^n:

$$\beta \mapsto h_{\phi(\beta)}(x) = \phi^{-1}(\phi(x) + \phi(\beta)) = \phi^{-1}(\phi(\beta) + \phi(x)) \in C^n(J)$$

for fixed $x = x_0 \in \mathbf{R}$, $\phi(J) = \mathbf{R}$.

Or, if we introduce $\alpha = c(\beta)$, where c is a discontinuous solution of

$$c(\beta_1 + \beta_2) = c(\beta_1) + c(\beta_2) , \quad \text{see [1, p. 35]} ,$$

then $\beta \mapsto h_{c(\beta)}(x)$ is not continuous, but $h_{c(\beta)}$ is still an iteration group with respect to β.

However, if we require both that $\beta \mapsto h_{p(\beta)}$ be an iteration group and that $\beta \mapsto h_{p(\beta)}$ remain at least continuous as $\alpha \mapsto h_\alpha$ was, then

$$p(\beta) = k\beta\,, \quad k \text{ being a constant}\,,$$

(see [1, p. 34](, and the smoothness of $\beta \mapsto h_{p(\beta)}$ is exactly the same, as that of $\alpha \mapsto h_\alpha$.

Anyway, the smoothness of H^* does not depend on parametrization of α in $\alpha \mapsto h_\alpha$.

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