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FACTORIZATIONS OF MATRICES AND FUNCTIONS  
OF TWO VARIABLES

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In this paper we shall give a characterization of functions and matrices that can be decomposed in the forms

$$h(x, t) = \sum_{k=1}^n f_k(x) g_k(t), \quad \text{and} \quad (a_{ij}) = \left( \sum_{k=1}^n b_k(i) c_k(j) \right).$$

In the case when  $h$  is sufficiently many times differentiable we get a characterization and a construction of  $f_k$  and  $g_k$  from  $h$  in terms of partial and ordinary differential equations.

Without regularity conditions on the function  $h$  and for matrices, we give a characterization and even explicit formulas for evaluating  $f_k, g_k,$  and  $b_k(i), c_k(j)$ . These formulas enable us to perform efficient computer computations, because the values of  $f_k(x), g_k(t)$ , as well as  $b_k(i), c_k(j)$  can be evaluated parallelly for different  $(x, t)$ , and  $(i, j)$  by pointwise multiplications only.

Moreover, if  $h$  is continuous or of a class  $C^d$  on  $I \times J$ , then the same kind of regularity holds for  $f_k$  on  $I$  and  $g_k$  on  $J$  for all  $k = 1, \dots, n$ . This means that we also have explicit formulas for solutions of the partial and ordinary differential equations mentioned above.

We write

$$D_m(h) := \begin{vmatrix} h & h_t & h_{tt} & \dots & h_{t^m} \\ h_x & h_{xt} & h_{xtt} & \dots & h_{xt^m} \\ \dots & \dots & \dots & \dots & \dots \\ h_{x^m} & h_{x^m t} & h_{x^m t t} & \dots & h_{x^m t^m} \end{vmatrix}$$

for a function  $h$  of  $x$  and  $t$  with continuous  $\partial^{i+j} h / \partial^i x \partial^j t = h_{x^i t^j}$  on  $I \times J \subset \mathbb{R}^2$ ,  $i, j \leq m$ . (Here  $I$  and  $J$  are unions of intervals.)

**Theorem 1.** *If a function  $h : I \times J \rightarrow \mathbb{R}$ , having continuous derivatives  $h_{x^i t^j}$  for  $i, j \leq n$ , can be written in the form*

$$(1) \quad h(x, t) = \sum_{k=1}^n f_k(x) g_k(t) \quad \text{on} \quad I \times J,$$

then

$$(2) \quad \det D_n(h) \equiv 0 \quad \text{on } I \times J.$$

If, moreover,  $f_k \in C^n(I)$ ,  $g_k \in C^n(J)$  and

$$(2_1) \quad \det (f_k^{(j)}(x)) \neq 0 \quad \text{for all } x \in I \quad \text{and} \quad \det (g_k^{(j)}(t)) \neq 0 \quad \text{for all } t \in J,$$

then also

$$(2_2) \quad \det D_{n-1}(h) \neq 0 \quad \text{for all } (x, t) \in I \times J$$

holds.

If  $h$  satisfies (2) and (2<sub>2</sub>) then there exist  $f_k \in C^n(I)$  and  $g_k \in C^n(J)$ ,  $k = 1, \dots, n$ , such that (1) and (2<sub>1</sub>) hold (and thus  $f_k$  and  $g_k$  are linearly independent). All decompositions of  $h$  of the form

$$h(x, t) = \sum_{k=1}^n \bar{f}_k(x) \bar{g}_k(t)$$

are exactly those for which

$$(\bar{f}_1, \dots, \bar{f}_n) = (f_1, \dots, f_n) \cdot C^T, \quad \text{and} \quad (\bar{g}_1, \dots, \bar{g}_n) = (g_1, \dots, g_n) \cdot C^{-1},$$

where  $C$  is an arbitrary  $n$  by  $n$  nonsingular constant matrix,  $C^T$  and  $C^{-1}$  being its transpose and inverse, respectively.

**Remark 1.** The functions  $f_k$  and  $g_k$  in (1) can be constructed from an  $h$  satisfying (2) and (2<sub>2</sub>) as solutions of two ordinary linear differential equations with coefficients evaluated from  $h$  (see (4<sub>1</sub>) below for  $f_k$  and its transpose analogue for  $g_k$ ). For constructing  $f_k$  and  $g_k$  from  $h$  satisfying (2) and (2<sub>1</sub>), see also Theorem 3 and Remark 5 below.

Proof of Theorem 1. If  $h$  is of the form (1), then  $h(x, t_0), h_t(x, t_0), \dots, h_{t^n}(x, t_0)$  are  $n + 1$  functions, each of them is a linear combination with constant coefficients of  $n$  functions  $f_1, \dots, f_n$ , so  $h, h_t, \dots, h_{t^n}$  are linearly dependent. Hence their Wronski determinant of order  $n + 1$  is zero, i.e.  $\det D_n(h) = 0$ . We also have

$$(3) \quad D_{n-1}(h) = \begin{vmatrix} \sum f_k g_k & \sum f_k g'_k & \dots & \sum f_k g_k^{(n-1)} \\ \sum f'_k g_k & \sum f'_k g'_k & \dots & \sum f'_k g_k^{(n-1)} \\ \dots & \dots & \dots & \dots \\ \sum f_k^{(n-1)} g_k & \sum f_k^{(n-1)} g'_k & \dots & \sum f_k^{(n-1)} g_k^{(n-1)} \end{vmatrix} = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f'_1 & f'_2 & \dots & f'_n \\ \dots & \dots & \dots & \dots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \cdot \begin{vmatrix} g_1 & g'_1 & \dots & g_1^{(n-1)} \\ g_2 & g'_2 & \dots & g_2^{(n-1)} \\ \dots & \dots & \dots & \dots \\ g_n & g'_n & \dots & g_n^{(n-1)} \end{vmatrix},$$

hence (2<sub>2</sub>) holds because of (2<sub>1</sub>).

If  $h$  satisfies (2) and (2<sub>2</sub>), then there exist  $A_i = A_i(x, t)$ ,  $i = 0, \dots, n-1$ , such that

$$(4) \quad \begin{aligned} h_{t^n} &= A_0 h + A_1 h_t + \dots + A_{n-1} h_{t^{n-1}} \\ h_{t^n x} &= A_0 h_x + A_1 h_{tx} + \dots + A_{n-1} h_{t^{n-1}x} \\ &\dots \\ h_{t^n x^n} &= A_0 h_{x^n} + A_1 h_{tx^n} + \dots + A_{n-1} h_{t^{n-1}x^n} \end{aligned}$$

holds. These  $A_i$  are differentiable with respect to  $x$ , because the system of the first  $n$  equations of (4) has a (unique) system of solutions  $A_0, \dots, A_{n-1}$  which are quotients of polynomials in  $h_{x^i t^j}$ ,  $i \leq n-1, j \leq n$ . By differentiating (4<sub>1</sub>) with respect to  $x$  and subtracting (4<sub>2</sub>) we get

$$\frac{\partial A_0}{\partial x} h + \frac{\partial A_1}{\partial x} h_t + \dots + \frac{\partial A_{n-1}}{\partial x} h_{t^{n-1}} = 0.$$

Similarly (4<sub>2</sub>) and (4<sub>3</sub>) give

$$\frac{\partial A_0}{\partial x} h_x + \frac{\partial A_1}{\partial x} h_{xt} + \dots + \frac{\partial A_{n-1}}{\partial x} h_{x t^{n-1}} = 0$$

and analogously up to

$$\frac{\partial A_0}{\partial x} h_{x^{n-1}} + \frac{\partial A_1}{\partial x} h_{x^{n-1}t} + \dots + \frac{\partial A_{n-1}}{\partial x} h_{x^{n-1}t^{n-1}} = 0.$$

Since  $\det D_{n-1}(h) \neq 0$ , all  $\partial A_i / \partial x = 0$ . Hence  $A_i$  are functions of  $t$  only.

Thus  $h$  satisfies (4<sub>1</sub>) for all  $x_0 \in I$  and it can be written as

$$(5) \quad h(x_0, t) = \sum_{k=1}^n f_k(x_0) g_k(t),$$

where  $g_k$  are independent solutions of (4<sub>1</sub>); hence also  $g_k \in C^n(J)$  and  $\det(g_k^{(j)}) \neq 0$  on  $J$ , cf. (2<sub>1</sub>). Any other set  $\bar{g}_k$  of independent solutions of (4<sub>1</sub>) satisfies

$$(6) \quad (\bar{g}_1, \dots, \bar{g}_n) = (g_1, \dots, g_n) C^{-1}$$

with a nonsingular constant  $n$  by  $n$  matrix  $C$ .

Now, from (5) we have  $f_k \in C^n(I)$  and due to (3), where  $\det D_{n-1}(h) \neq 0$ , the first part of (2<sub>1</sub>) holds too and  $f_k$  are independent (since they also satisfy a linear differential equation). Moreover, after the  $g_k$  were chosen, the  $f_k$  are uniquely determined for a given  $h$  in order that (3) be satisfied. That also gives  $(\bar{f}_1, \dots, \bar{f}) = (f_1, \dots, f_n) \cdot C^T$ , if the  $\bar{g}_k$  are chosen as in (6). Q.E.D.

**Theorem 2.** A matrix  $H = (H_{ij})$ ,  $i = 1, \dots, r$ ;  $j = 1, \dots, s$ , can be written in the form

$$(7) \quad (H_{ij}) = \left( \sum_{k=1}^n F_k(i) G_k(j) \right)$$

with  $n$  independent vectors  $F_1(i), \dots, F_n(i)$ , and  $n$  independent vectors  $G_1(j), \dots, G_n(j)$ , if and only if

$$\text{rank } H = n, \quad n \leq \min(r, s).$$

If this assumption is satisfied, then all decompositions of  $H$  in the form

$$(H_{ij}) = \left( \sum_{k=1}^n \bar{F}_k(i) \bar{G}_k(j) \right)$$

are exactly those for which

$$\bar{F} = F \cdot C, \quad \bar{G} = C^{-1} \cdot G,$$

$C$  being a nonsingular real  $n$  by  $n$  matrix,

$$\bar{F} = (\bar{F}_{ik}) := (\bar{F}_k(i)), \quad F = (F_{ik}) := (F_k(i)),$$

$$\bar{G} = (\bar{G}_{kj}) := (\bar{G}_k(j)), \quad G = (G_{kj}) := (G_k(j)).$$

**Remark 2.** If the linear independence of  $F_k$  and  $G_k$  is not supposed, then  $\text{rank } H \leq n \leq \min(r, s)$ .

**Remark 3.** If the assumption of Theorem 2 is satisfied, then all the decompositions (7) can be constructed from  $H$  by using (8) below.

Proof of Theorem 2. 1° If (7) is satisfied, i.e.,  $H = F \cdot G$ , then  $\text{rank } F \leq n$ ,  $n \leq r$ ,  $\text{rank } G \leq n$ ,  $n \leq s$ , hence  $\text{rank } H \leq n$ ,  $n \leq \min(r, s)$ , cf. Remark 2. If  $F_k$  and  $G_k$  are linearly independent, then  $\text{rank } F = n$ ,  $\text{rank } G = n$ , and  $\text{rank } H = n$ .

2° Let  $\text{rank } H = n$ . Then  $H$  can be reindexed in the form

$$H = \begin{vmatrix} H^* & H^* \cdot M \\ P \cdot H^* & P \cdot H^* \cdot M \end{vmatrix}$$

where  $H^*$  is a nonsingular  $n$  by  $n$  matrix,  $M$  and  $P$  are suitable  $n$  by  $(s - n)$  and  $(r - n)$  by  $n$  matrices, respectively.

Choose any nonsingular  $n$  by  $n$  matrix  $F^*$  and put  $G^* := F^{*-1} \cdot H^*$ . Evidently  $H^* = F^* \cdot G^*$  and any such relation for a given  $H^*$  can be established exactly by taking  $F^* \cdot C$  and  $C^{-1} \cdot G^*$  instead of  $F^*$  and  $G^*$ , respectively, for any nonsingular  $n$  by  $n$  matrix  $C$ . Then we can write

$$(8) \quad H = \begin{vmatrix} F^* & \\ P \cdot F^* & \end{vmatrix} \cdot \|G^*, G^* \cdot M\|,$$

or

$$H = F \cdot G \quad \text{for} \quad F := \begin{vmatrix} F^* & \\ P \cdot F^* & \end{vmatrix}, \quad G := \|G^*, G^* \cdot M\|.$$

All factorizations of  $H$  into  $\bar{F} \cdot \bar{G}$  are exactly those where

$$\bar{F} = \left\| \begin{array}{c} F^* \cdot C \\ P \cdot F^* \cdot C \end{array} \right\| = F \cdot \bar{C}, \quad \text{and} \quad \bar{G} = \|C^{-1} \cdot G^*, C^{-1} \cdot G^* \cdot M\| = C^{-1} \cdot G,$$

Q.E.D.

Now we shall apply Theorem 2 to get a characterization of functions  $h$  satisfying (1) without requiring the differentiability of  $h$ .

According to [1, Sec. 4.2.5],  $n$  functions  $\phi_k : S \rightarrow R$ , ( $S$  is a subset of  $R$ ,  $S$  need not be an interval;  $k = 1, \dots, n$ ) are linearly independent on  $S \subset R$ , if there exist  $n$  points  $x_k$ ,  $k = 1, \dots, n$ , in  $S$  such that the matrix

$$W[\phi_k, x_k]_{k=1}^n = \left\| \begin{array}{c} \phi_1(x_1), \dots, \phi_n(x_1) \\ \phi_1(x_n), \dots, \phi_n(x_n) \end{array} \right\|$$

is regular.

**Theorem 3.** *Let  $I$  and  $J$  be arbitrary nonempty sets.*

*A function  $h : I \times J \rightarrow R$  can be written in the form (1) with linearly independent  $f_k$  and  $g_k$  if and only if the maximum of the rank of the matrices*

$$(h(x_i, t_j)), \quad i = 1, \dots, r; \quad j = 1, \dots, s,$$

*is  $n$  when  $x_i \in I$ ,  $t_j \in J$ ,  $r$  and  $s$  being arbitrary integers.*

*If, in addition,  $I$  and  $J$  are intervals,  $h \in C^d(I \times J)$ ,  $d \geq 0$ , then  $f_k \in C^d(I)$  and  $g_k \in C^d(J)$  for all  $k = 1, \dots, n$ .*

**Remark 4.** The explicit formula for evaluating  $f_k$  and  $g_k$  from a given  $h$  will be given in (10) below. From the point of view of computation it is important that, when constant matrices  $H^*$  and  $G^*$  are chosen, then the values of  $f_k$  at  $x$  and  $g_k$  at  $t$  depend only on the values of  $h$  at  $(x_k, t)$  and at  $(x, t_k)$ ,  $k = 1, \dots, n$ , respectively. Hence the  $f_k$  and  $g_k$  can be evaluated separately and at the same time for different arguments, see (10).

**Remark 5.** Note that the formula (10) below also gives in a constructive way the functions  $f_k$  and  $g_k$  by which the solutions  $h$  of the nonlinear partial differential equation  $\det D_n(h) = 0$  can be decomposed in the sense of (1) without the necessity of solving linear differential equations as mentioned in Remark 1.

Proof of Theorem 3. 1° If  $h(x, t) = \sum_{k=1}^n f_k(x) g_k(t)$  on  $I \times J$ , then (7) is satisfied for

$$H_{ij} := h(x_i, t_j), \quad F_k(i) := f_k(x_i), \quad G_k(j) := g_k(t_j)$$

and any  $r$ -tuple of  $x_i \in I$  and  $s$ -tuple of  $t_j \in J$ . In view of Remark 2,  $\text{rank}(H_{ij}) \leq n$ . Since  $f_k$  and  $g_k$  are linearly independent, there exist an  $n$ -tuple of  $x_i \in I$  and an  $n$ -tuple

of  $t_j \in J$  such that both  $W[f_k, x_k]$  and  $W[g_k, t_k]$  are nonsingular. For these  $n$ -tuples,  $(H_{ij}) = W[f_k, x_k] \cdot W^T[g_k, t_k]$ , hence the rank  $n$  is achieved.

2° Let  $H^* := (H_{ij}^*) = (h(x_i, t_j))$ ,  $i = 1, \dots, n$ ;  $j = 1, \dots, n$ , be a nonsingular  $n$  by  $n$  matrix. Consider an  $(n + 1)$  by  $(n + 1)$  matrix

$$\begin{vmatrix} h(x_1, t_1), \dots, h(x_1, t_n), h(x_1, t) \\ h(x_2, t_1), \dots, h(x_2, t_n), h(x_2, t) \\ \dots \\ h(x_n, t_1), \dots, h(x_n, t_n), h(x_n, t) \\ h(x, t_1), \dots, h(x, t_n), h(x, t) \end{vmatrix} = \begin{vmatrix} H^* & a \\ b & c \end{vmatrix},$$

where  $a$  is an  $n$  by 1 vector, and  $b$  is a 1 by  $n$  vector. The rank of the matrix is  $n$ .

Choose any nonsingular  $n$  by  $n$  matrix  $C$ . Then

$$(9) \quad \begin{vmatrix} H^* \cdot C \\ b \cdot C \end{vmatrix} \cdot \|C^{-1}, C^{-1} \cdot H^{*-1} \cdot a\| = \begin{vmatrix} H^* & a \\ b & c \end{vmatrix},$$

see also (8). Define

$$(f_1(x), \dots, f_n(x)) := (h(x, t_1), \dots, h(x, t_n)) \cdot C = b \cdot C,$$

$$(10) \quad \begin{vmatrix} g_1(t) \\ \dots \\ g_n(t) \end{vmatrix} := C^{-1} \cdot H^{*-1} \cdot \begin{vmatrix} h(x_1, t) \\ \dots \\ h(x_n, t) \end{vmatrix} = C^{-1} \cdot H^{*-1} \cdot a$$

$x \in I$ ,  $t \in J$ . Due to (9),

$$c = b \cdot C \cdot C^{-1} \cdot H^{*-1} \cdot a = b \cdot H^{*-1} \cdot a,$$

and we have

$$h(x, t) = \sum_{k=1}^n f_k(x) g_k(t).$$

The matrices  $C$  and  $H^*$  in (10) are constant, hence, if  $h \in C^d(I \times J)$ ,  $d \geq 0$ ,  $I$  and  $J$  are intervals, then

$x \mapsto h(x, t) \in C^d(I)$  and  $t \mapsto h(x, t) \in C^d(J)$ , and also  $f_k \in C^d(I)$ ,  $g_k \in C^d(J)$  for all  $k = 1, \dots, n$ . Q.E.D.

A program for evaluating  $f_k$  and  $g_k$  from a given  $h$  may be constructed as follows (dot denotes matrix multiplication):

STEP 1 For a sufficiently large or dense set  $\{(x_i, t_j); i = 1, \dots, s; j = 1, \dots, r\}$ , determine the rank of the matrix  $H = (h(x_i, t_j))$  as  $n$ , find a regular  $n$  by  $n$  submatrix  $H^*$  of  $H$ , the corresponding indices forming the sets  $K$  and  $L$  (each of the cardinality  $n$ ).

STEP 2 Choose a regular  $n$  by  $n$  matrix  $C$ .

STEP 3(i) Take the row vector  $\{h(x_i, t_l); l \in L\}$  from  $H^*$  and determine the vector  $(f_1(x_i), \dots, f_n(x_i)) := \{h(x_i, t_l); l \in L\} \cdot C$  for  $i = 1, 2, \dots, s$ .

STEP 4(j) Take the column vector  $\{h(x_k, t_j); k \in K\}$  from  $H^*$  and determine the

$$\text{vector } \begin{pmatrix} g_1(t_j) \\ \dots \\ g_n(t_j) \end{pmatrix} := \tilde{C}^{-1} \cdot H^{*-1} \cdot \{h(x_k, t_j); k \in K\} \text{ for } j = 1, 2, \dots, r.$$

STEP 5 Check the relation

$$(f_1(x_i), \dots, f_n(x_i)) \cdot \begin{pmatrix} g_1(t_j) \\ \dots \\ g_n(t_j) \end{pmatrix} = h(x_i, t_j).$$

For  $i = 1, \dots, s$  and  $j = 1, \dots, r$  it should be satisfied, otherwise there is an error in the computation.

STEP 6 Now, we may enlarge the initial set  $\{(x_i, t_j); i = 1, \dots, s; j = 1, \dots, r\}$  by adding  $(x_i, t_j)$ ,  $i > s$  and/or  $j > r$ . Go to STEP 3(i),  $i > s$ , and STEP 4(j),  $j > r$ . Then do STEP 5 for  $i > s, j > r$ . If all the relations are satisfied, then the extensions of  $f_k(x_i)$  for  $i > s$  and  $g_k(t_j)$  for  $j > r$  form the factorization (1). If there is an error here, we may either accept the extended  $f_k$  and  $g_k$  as approximations of our factorization, or we may change the initial set of  $(x_i, t_j)$ , or enlarge it by adding points with  $i > s, j > r$ .

**Remark 6.** STEP 3 and STEP 4 are independent of each other, and also STEP 3(i) and STEP 3(i') as well as STEP 4(j) and STEP 4(j') are independent, hence they can be performed simultaneously.

**Remark 7.** All arithmetic operations are expressible by pointwise multiplications only.

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#### References

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