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CONGRUENCE EXTENSION FROM A SEMILATTICE TO THE FREELY GENERATED DISTRIBUTIVE LATTICE

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The author has shown that every congruence for the \wedge -operation of a semilattice which is also compatible for any set of distributive \vee 's is induced by a lattice congruence on the distributive lattice freely generated with preservation of these sups [F, Theorem DL]. Cornish and Hickman [CH], and in more detail Hickman [H], have addressed the question of uniqueness of the inducing congruence, basing themselves on a preliminary analysis of the structure of these congruences. It will be pointed out here that their main results may be obtained more simply, and in a considerably more general setting, by a systematic use of the author's result; moreover an explicit form for the smallest inducing congruence may in their setting be deduced from MacNeille's construction [M] of the freely generated lattice — a construction to which they have inadvertently had recourse for their study of congruences.

For example, the main result of [CH], that uniqueness obtains in the presence of "the upper bound property", may be seen rather quickly by remembering that the lattice L consists of \vee 's of families in the semilattice S . Then if one attempts to identify $\bigvee a_i$ with $\bigvee b_j$ in L one will be forced to identify a_i with $a_i \wedge \bigvee b_j = \bigvee_j a_i \wedge b_j$ which, being bounded by $a_i \in S$, is $= c_i \in S$; similarly $b_j \sim \bigvee_i a_i \wedge b_j = d_j \in S$; and of course $\bigvee c_i = \bigvee d_j$ in L . But conversely these conditions require the identification of $\bigvee a_i$ with $\bigvee b_j$ by any lattice congruence inducing \sim in S , and therefore such a congruence is uniquely determined. Unlike the development in [CH], the \vee 's occurring here are not limited to be finite, nor has it been necessary to postulate that all existing (finite) \vee 's in S be distributive, as will be elaborated more fully below.

Again, the (at first sight unexpected) implication i) \rightarrow ii) of [H, Theorem 2.4], to the effect that non-uniqueness of the inducing lattice congruence entails that restriction to S does not preserve the join of congruences in L (whose proof in [H] takes up two and a half pages) drops out as follows: By dividing modulo a congruence \sim on L which extends non-uniquely to L one reduces to a non-identical congruence \sim on S inducing the identity on S (this type of reduction will also be detailed more fully); say it identifies the pair $\bigvee a_i < \bigvee b_j$ where some $b_j = b \not\leq \bigvee a_i$. According to the form of the principal congruence generated by a comparable pair in

a distributive lattice [G, Theorem 3 p. 74], $\theta(a \wedge b, \bigvee a_i)$ for any $a = a_i$ fails to identify a and b while $\theta(\bigvee a_i, \bigvee b_j) \subset \sim$ induces the identity on S , whence the join of their restrictions fails to identify a and b — but their join in L , $\theta(a \wedge b, \bigvee b_j)$, does. This, too, will go through in a more general setting; we turn now to spelling out this generalization.

Recall that a (not necessarily finite) sup $\bigvee a_i$ in a semilattice S is *distributive* if for each a the sup of the translate by a of the a_i , $\bigvee a \wedge a_i$, also exists and $= a \wedge \bigvee a_i$.¹⁾ MacNeille [M] has shown (an outline is sketched in [F']) that every \vee -semilattice can be embedded in a complete, completely join-distributive lattice with preservation of all its distributive \vee 's. It follows that for any collection D of subsets having distributive \vee 's, the universal morphism or reflection (into the subcategory of complete \vee -preserving morphisms) which sends S into the complete such lattice L required to preserve only the \vee 's in D , is also an embedding. Note that the complete join-distributivity of L entails that it consists of the \vee 's (in L) of subsets of S (i.e. S is \vee -generating or “join-dense”), the pairwise \wedge being calculated as $(\bigvee a_i) \wedge (\bigvee b_j) = \bigvee (a_i \wedge b_j)$. S is a fortiori join-dense in every intermediate \wedge -semilattice $S \subset \subset F \subset L$; and the inclusion into F is universal among semilattice morphisms S' from S which preserve the \vee 's in D as well as send some of its subsets, whose \vee 's include the elements of F , (these subsets may be taken closed for pairwise \wedge 's) to subsets having distributive \vee 's: Indeed for any such morphism the image semilattice S' may be embedded into a complete L' with preservation of the \vee 's of both the images of the subsets in D and of any of those whose \vee 's include all the elements of F ; and then the given semilattice morphism, which may even be extended to send L into L' in complete \vee -preserving fashion,²⁾ will have a restriction to F which sends it into S' .

It was shown in [F, Theorem DL] that every congruence for \wedge compatible also for the \vee 's in a translation-closed class D (in the “weak” sense that from $a_i \sim a'_i$ and the presence of both subsets in D follows $\bigvee a_i \sim \bigvee a'_i$) is the restriction of a lattice congruence on F — this was done there for the case that D consists only of pairwise \vee 's in S and F of the \vee 's of its finite subsets in L , but the result holds in general: the quotient map modulo the given congruence \sim sends the \vee 's in D to distributive \vee 's in the quotient semilattice³⁾ whence its composition with the embedding of the quotient into the complete lattice universal for preserving these image \vee 's extends to a complete \vee -preserving morphism on L whose kernel meets any intermediate F in a congruence extending \sim . It is in fact the smallest on F , compatible for the \vee 's

¹⁾ This identity for any (partial) operation $\vee \geq$ its arguments already shows it to be sup: for if $a_i = a \wedge a_i$ then $\bigvee a_i = a \wedge \bigvee a_i$.

²⁾ Thus L is also the universal, complete \vee -preserving, completion of F for preservation of D augmented with such a collection of $(L-)$ \vee 's of S -subsets (they are distributive in F).

³⁾ Use footnote 1; the structure of a translation-closed class of \vee 's is preserved by homomorphic image, product and subsemilattice.

of any translation-closed class including D and yielding all of F , which induces \sim on S , as follows from the universality noted on the preceding page of the inclusion of S/\sim into the image of F .

An explicit form for this smallest congruence could therefore be derived by carrying back to F a description of equality in such complete join-distributive lattices universal over the quotient semilattice. Now such a description appears in MacNeille [M] at the beginning of Section 12 p. 446 (one may also consult [F']) for the case D all distributive \vee 's in S and $F = L$; but analysis of his argument shows that it remains valid, thus furnishing the desired description, if only D is translation closed, includes every subset having a greatest element, and satisfies the following "Fubini" condition: if a doubly-indexed a_{ij} has $\bigvee_j a_{ij}$, $\bigvee_i a_{ij}$ and $\bigvee_j (\bigvee_i a_{ij})$ all in D then it also has $\bigvee_i (\bigvee_j a_{ij})$ ⁴). Specifically, let such a collection D of subsets having distributive \vee 's in S be specified. Then on the power set of S (construed as indexed subsets of S) the relation: for every j , $\{a_i \wedge b_j\}_i \in D$ and $\bigvee_i a_i \wedge b_j = b_j$, is reflexive and transitive. Modulo the equivalence making this a partial order the image of every subset is the join of its singleton subset images, whence the quotient is a complete lattice. The images of singletons are an order-isomorphic copy of S , which permits writing every element of the lattice (not necessarily uniquely) as $\bigvee a_i$, inasmuch as the \vee 's in D are preserved by the embedding of S (in fact they are the only singleton \vee 's of singletons). The finite \vee 's in S are preserved by virtue of $(\bigvee a_i) \wedge (\bigvee b_j) = \bigvee_{ij} a_i \wedge b_j$, from which also complete join-distributivity follows.

It does not appear to be particularly easy to assure the Fubini condition in a quotient. It would pass to the images, from a class of distributive \vee 's in which it holds, modulo an \wedge -congruence compatible with them in the "strong" sense that from $a_i \sim a'_i$ and the presence of one of the subsets in D follows that of the other (in addition to $\bigvee a_i \sim \bigvee a'_i$). It also holds in the class of all subsets less numerous than some regular cardinal which have distributive \vee 's in S , and so with D just these subsets augmented with the subsets having greatest elements, the above sufficient criterion for the description of equality in L is fulfilled. The augmentation is not needed for the description of equality in the sublattice F of \vee 's of such cardinal-limited subsets of S : This F realizes the lattice over S universal for preserving all the \vee 's in D and in which all limited subsets have distributive \vee 's. This is the situation treated by [CH] when the limited subsets are the finite ones (under the additional requirement that all existent finite \vee 's are distributive) and by [FS] more generally when they are those bounded by a regular cardinal — both treatments retrace MacNeille's.

Now when the images of the designated class D of subsets with distributive \vee 's in S fulfill the above criterion modulo \sim , an \wedge -congruence compatible for these \vee 's,

⁴ Only the presence in D of $\{\bigvee_j a_{ij}\}$ as an i -indexed subset is at issue: its \vee , $\bigvee_{ij} a_{ij}$, exists in S and (as a \vee which becomes distributive on decomposing its terms into distributive \vee 's) is distributive. Also, it is enough to have this for $a_{ij} = a_i \wedge b_j$.

then MacNeille's description of equality for \vee 's of subsets of S/\sim in the distributive lattice completion universal for preserving all these image \vee 's, yields that the smallest lattice congruence extending \sim to any distributive lattice universal for preserving the \vee 's in D identifies $\bigvee a_i$ with $\bigvee b_j$ (recall that this universal lattice, as a sublattice of L , consists of \vee 's of subsets of S) just when $a_i \wedge b_j \sim c_{ij} \in S$ with $a_i \sim \bigvee_j c_{ij} \in D$ (whence also $\sim \bigvee_j a_i \wedge c_{ij} \in D$) and symmetrically $b_j \sim \bigvee_i b_i \wedge d_{ji}$.

Some supplementary remarks: One can of course always reduce $\bigvee a_i \sim \bigvee b_j$ to $a_i \sim \bigvee a_i \wedge b_j$ (and symmetrically) by using that \sim is an \wedge -congruence and that all \vee 's in F are distributive, but the \vee on the right is in general only in F and so this does not, like MacNeille's, provide a description internal to S — unless one makes an additional postulate such as the "upper bound property" of [CH]: Cf the analysis of their theorem given at the beginning.

They have couched their discussion in terms of the operation of restriction of congruences from F to S (a surjection by the author's cited Theorem DL): this preserves arbitrary \cap (and so in particular there is a smallest inducing congruence) whence the lack of lattice preservation touches only the join of congruences. A dual discussion would assign to each congruence on S its smallest inducing congruence on F and lead to a complete join-preserving injection of the congruence lattice of S onto the sub-semilattice of smallest inducing congruences of F — but now it is presumably \cap which is not preserved. (The smallest congruence inducing \sim must identify $\bigvee a_i$ with $\bigvee a'_i$ whenever $a_i \sim a'_i$: this is already reflexive symmetric and substitutional but taking the transitive closure cannot be expected to commute with \cap .)

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