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A SIMPLE PROOF OF THE MINIMAX-THEOREM

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As it is well-known (see e.g. the references [1], ..., [5]) the Minimax-Theorem can be verified by using the Kakutani-fixed-point-Theorem or by applying the duality theory of convex optimization. This paper presents a simpler proof based in the main part on induction and on the continuity of the solutions of some parametric optimization problems. Further the Weierstrass Theorem related to the minimum of a continuous function is applied.

Theorem. *Let A and B be convex, compact and non-empty subsets of the Euclidean space E_n and let $f : A \times B \to R$ be a contiuous function which is convex on A for each fixed b in B and concave on B for each fixed a in A. Then the function f has at least one saddle point, i.e. a point (a, b) in $A \times B$ satisfying*

$$f(a, y) \leqq f(a, b) \leqq f(x, b) \tag{1}$$

for all x in A and y in B.

Proof. At first let us additionally assume that the function f is strongly convex-concave. Then we find that at most one saddle point exists and the functions $a(t)$, $b(t)$ and $x(t)$ defined below will be single-valued.

Obviously, the Theorem holds if the sum

$$d = \dim A + \dim B$$

is equal to zero. Now we consider the case $d = k$ and suppose the Theorem to be true for $d < k$. One of the sets A and B, say A, then contains more than one point and there is an element c in E_n such that A is not included in any affine half-space

$$C_t = \{x \mid x \in E_n, (c, x) = t\} \quad (t \in R).$$

Setting

$$A_t = A \cap C_t \quad \text{and} \quad T = \{t / A_t \neq \emptyset\}$$

we observe that T is a closed interval $[t_*, t^*]$ and that

$$\dim A_t < \dim A$$

for all t in T. By applying the theorem to A_t and B there is exactly one point $(a(t), b(t))$ in $A_t \times B$ satisfying

$$(1)_t \qquad f(a(t), y) \leqq f(a(t), b(t)) \leqq f(x, b(t))$$

for all $x \in A_t$, $y \in B$. Further, the points $a(t)$ and $b(t)$ continuously depend on t (see the remark) and this is also true for the points $x(t)$ which minimize the function $f(., b(t))$ on A.

If the inequality

$$(2) \qquad f(a(t), b(t)) \leqq f(x(t), b(t))$$

holds for some t in T then the point $(a(t), b(t))$ fulfils (1) and the proof (under our additional assumption) is complete. In the other case, however, one concludes from $(1)_t$

$$x(t) \in A \setminus A_t \quad \text{and} \quad (c, x(t)) \neq t$$

for all t in T. For the continuous function

$$h(t) = t - (c, x(t))$$

we thus obtain $h(t_*) < 0$ and $h(t^*) > 0$, and consequently a point t_0 in T exists such that $h(t_0) = 0$. That means $x(t_0) \in A_{t_0}$ and leads to a contradiction.

Hence the inequality (2) holds for some t in T and the Theorem is true for strongly convex-concave f.

In order to complete the proof for the full Theorem we introduce (for $\varepsilon > 0$) the strongly convex-concave function

$$f_\varepsilon(x, y) = f(x, y) + \varepsilon\|x\|^2 - \varepsilon\|y\|^2$$

where the Euclidean norm is taken. Since for each $\varepsilon > 0$ a saddle point $(a_\varepsilon, b_\varepsilon)$ exists with respect to f_ε we find a saddle point for f as a cluster point of any sequence $\{(a_\varepsilon, b_\varepsilon)\}_{\varepsilon \to 0}$.

Remark. The continuity of the functions $a(t)$, $b(t)$ and $x(t)$ considered above as well as the fact that any cluster point of the sequence $\{(a_\varepsilon, b_\varepsilon)\}_{\varepsilon \to 0}$ fulfils (1) follows from well-known stability results for parametric optimization problems. For completeness and convenience we add the following Lemma whose proof also shows that the continuity-properties can be verified without great investigations.

Lemma. *Let A, B and f as in the Theorem, let $g : A \times B \to R$ be continuous and c, d in E_n be arbitrary points. For $t \in R$ we form*

$$A_t = \{x \mid x \in A,\ (c, x) = t\}, \quad B_t = \{y \mid y \in B,\ (d, y) = t\}$$

and for $\varepsilon \in R$ we define $F_\varepsilon(x, y) = f(x, y) + \varepsilon \,.\, g(x, y)$. Then, the set M of all points (a, b, ε, t) such that (a, b) is a saddle point of F_ε with respect to $A_t \times B_t$ is closed in E_{2n+2}.

Proof. Since A and B are compact it sufficies to show that for any (a, b, ε, t) in $(A_t \times B_t \times E_2) \setminus M$ there is a neighbourhood N that does not meet M. Since the saddle point condition is not satisfied there is an $x \in A_t$ (or a corresponding point $y \in B_t$) such that

$$F_\varepsilon(x, b) < F_\varepsilon(a, b) .$$

By the continuity-assumptions there exists a $\delta > 0$ such that if $\max \{|\varepsilon' - \varepsilon|, \|x' - x\|, \|a' - a\|, \|b' - b\|\} < \delta$ we obtain

(3) $$F_{\varepsilon'}(x', b') < F_{\varepsilon'}(a', b').$$

Now we choose, if they exist, points x^+ and x^- in A with

$$(c, x^+) > t , \quad (c, x^-) < t$$

and, if one does not exist, we put the corresponding point x^+ or x^- equal to x. For sufficiently small $|t' - t|$ then either $A_{t'} = \emptyset$ holds or one of the line segments $[x, x^+]$, $[x, x^-]$ meets $A_{t'}$ where the common point $x_{t'}$ converges to x as $t' \to t$. Hence, we have either $A_{t'} = \emptyset$ or

$$A_{t'} \cap \{x' \mid \|x' - x\| < \delta\} \neq \emptyset$$

if $|t' - t|$ is small enough, say less than δ'.

Thus we obtain from (3) that M does not meet the set

$$N = \{(a', b', \varepsilon', t') \mid |t' - t| < \delta', \max \{\|a' - a\|, \|b' - b\|, |\varepsilon' - \varepsilon|\} < \delta\}$$

and the Lemma is verified.

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