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THE LEAST SEPARATIVE CONGRUENCE ON A WEAKLY COMMUTATIVE SEMIGROUP

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In [1], a relation \( \pi \) on an arbitrary semigroup \( S \) has been defined. For elements \( a \) and \( b \) of \( S \), \( a \pi b \) if and only if \( ab^n = b^{n+1} = b^n a \) and \( ba^n = a^{n+1} = a^n b \) for a positive integer \( n \). It has been proved that, if \( S \) is a weakly commutative semigroup, then \( \pi \) is a separative congruence on \( S \). The author has proved that, if \( S \) is a duo semigroup (i.e. every one-sided ideal of \( S \) is two-sided), then \( S/\pi \) is a maximal separative homomorphic image of \( S \). See Theorem 5 of [1].

In this note we shall extend this result on duo semigroups to weakly commutative semigroups.

**Definition 1.** A semigroup \( S \) is called **weakly commutative** if, for any \( a, b \in S \), we have \((ab)^k = xa = by\) for some \( x, y \in S \) and a positive integer \( k \). See Definition 6.4 of [2].

**Definition 2.** We define a relation \( \pi \) on a semigroup \( S \) as follows: \( a \pi b \) if and only if \( ab^n = b^{n+1} = b^n a \) and \( ba^n = a^{n+1} = a^n b \) for a positive integer \( n \). See [1].

**Remark 1.** Let \( S \) be a semigroup, \( a \) and \( b \in S \) and \( \varrho \) a congruence on \( S \). If \( ab^{n+1} \varrho b^{n+2} \) and \( (ab^n)^m \varrho (b^{n+1})^m \) for positive integers \( n \) and \( m \), then \( (ab^n)^M \varrho (b^{n+1})^M \) for any positive integer \( M > m \).

Similarly, if \( b^{n+1} \varrho b^{n+2} \) and \( (b^n)^m \varrho (b^{n+1})^m \) for positive integers \( n \) and \( m \), then \( (b^n)^M \varrho (b^{n+1})^M \) for any positive integer \( M > m \).

**Proof.** We prove only the first part of the remark, because the second part can be proved in a similar way. Let us suppose \( ab^{n+1} \varrho b^{n+2} \), \( (ab^n)^m \varrho (b^{n+1})^m \) for some \( a, b \in S \) and positive integers \( n \) and \( m \). Let \( M \) be an arbitrary positive integer with \( M > m \). Then \( (ab^n)^M = (ab^n)^M \varrho (ab^n)^M \varrho (b^{n+1})^m = (ab^n)^M \varrho (ab^n)^M \varrho (b^{n+1})^m = \cdots \varrho (b^{n+1})^m \varrho (b^{n+1})^m = (b^{n+1})^M \).

**Lemma 1.** (B. Pondělíček [1]). If \( S \) is a weakly commutative semigroup, then \( \pi \) is a separative congruence on \( S \).

630
**Theorem 1.** If $S$ is a weakly commutative semigroup, then $S/\pi$ is a maximal separative homomorphic image of $S$.

**Lemma 2.** Let $S$ be a weakly commutative semigroup and $\varrho$ a separative congruence on $S$. Let $a, b \in S$. If $ab^n \varrho b^{n+1} \varrho b^n a$ and $ba^n \varrho a^{n+1} \varrho a^n b$ for a positive integer $n$, then $a \varrho b$.

**Proof.** Since $\varrho$ is a separative congruence, the result is true for $n = 1$. Assume now that the assertion holds for $n \geq 1$. Let $ab^{n+1} \varrho b^{n+2} \varrho b^{n+1} a$ and $ba^{n+1} \varrho a^{n+2} \varrho a^{n+1} b$. Since $S$ is weakly commutative, $(ab^n)^k = by$ for some $y \in S$ and a positive integer $k$. Thus $(ab^n)^{k+1} = ab^{n+1} y \varrho b^{n+2} y = b^{n+1} (ab^n)^k = b^{n+1} ab^n (ab^n)^{k-1} \varrho \varrho (b^{n+1})^2 (ab^n)^{k-1} \varrho \ldots \varrho (b^{n+1})^k b^{n+1}$ and $(b^n a)^t = ub$ for some $u \in S$ and a positive integer $t$. Thus

$$(b^n a)^{t+1} = ub^{n+1} a \varrho ub^{n+2} = (b^n a)^t b^{n+1} = (b^n a)^{t-1} b^n a b^{n+1} \varrho \varrho (b^n a)^{t-1} (b^{n+1})^2 \varrho \ldots \varrho (b^{n+1})^t b^{n+1}.$$

Consequently, $(ab^n)^{k+1} \varrho (b^{n+1})^{k+1}$ and $(b^{n+1})^{t+1} \varrho (b^n a)^{t+1}$. By Remark 1, it follows that $(ab^n)^m \varrho (b^{n+1})^m \varrho (b^n a)^m$ for a positive integer $m$.

Let $m_1 = \min \{ m : (ab^n)^m \varrho (b^{n+1})^m \varrho (b^n a)^m \}$. We prove that $m_1 = 1$. Let us suppose that $m_1 \neq 1$ and let

$$m_2 = \begin{cases} m_1 & \text{if } m_1 \text{ is an even number}, \\ m_1 + 1 & \text{if } m_1 \text{ is an odd number}. \end{cases}$$

Then, by Remark 1,

$$(ab^n)^{m_2} \varrho (b^{n+1})^{m_2} \varrho (b^n a)^{m_2}.$$ 

Let $m_3 = m_2/2$. Then $m_3 > m_1$ and

$$(ab^n)^{m_3} = (ab^n)^{m_2} = (b^{n+1})^{m_2} = ((b^{n+1})^{m_2})^2 =$$

$$= (b^{n+1})^{m_2} \varrho (b^n a)^{m_2} = (b^n a)^{2m_3} = ((b^n a)^{m_3})^2.$$

Moreover,

$$(ab^n)^{m_3} (b^{n+1})^{m_3} = (ab^n)^{m_2-1} ab^n b^{n+1} (b^{n+1})^{m_3-1} \varrho$$

$$\varrho (ab^n)^{m_3-1} b^{n+1} (b^{n+1})^{m_3-1} = (ab^n)^{m_3-1} (b^{n+1})^{m_3+1} \varrho \ldots \varrho (b^{n+1})^{2m_3} = ((b^{n+1})^{m_3})^2$$

and

$$(b^n a)^{m_3} (b^{n+1})^{m_3} = (b^n a)^{m_3-1} b^n a b^{n+1} (b^{n+1})^{m_3-1} \varrho$$

$$\varrho (b^n a)^{m_3-1} (b^{n+1})^{m_3-1} = (b^n a)^{m_3-1} (b^{n+1})^{m_3+1} \varrho \ldots \varrho (b^{n+1})^{2m_3} = ((b^{n+1})^{m_3})^2.$$

Thus we have $((ab^n)^{m_3})^2 \varrho (ab^n)^{m_3} (b^{n+1})^{m_3} \varrho ((b^{n+1})^{m_3})^2$ and

$$((b^n a)^{m_3})^2 \varrho (b^n a)^{m_3} (b^{n+1})^{m_3} \varrho ((b^{n+1})^{m_3})^2.$$ 

Since $\varrho$ is a separative congruence, it follows that

$$(ab^n)^{m_3} \varrho (b^{n+1})^{m_3} \varrho (b^n a)^{m_3}.$$ 

631
Since this result contradicts $m_3 < m_1$, we have $m_1 = 1$. Consequently, $ab^n \equiv b^{n+1} \equiv b^*a$. We can prove $ba^n \equiv a^{n+1} \equiv a^*b$ in a similar way. Hence we get $a \equiv b$. The result therefore follows by induction. Thus the lemma is proved.

The proof of Theorem 1. Let $\equiv$ be an arbitrary separative congruence on a weakly commutative semigroup $S$. If $a \equiv b$ $(a, b \in S)$, then $ab^n = b^{n+1} = b^*a$ and $ba^n = a^{n+1} = a^*b$ for a positive integer $n$. Thus $ab^n \equiv b^{n+1} \equiv b^*a$ and $ba^n \equiv a^{n+1} \equiv a^*b$. By Lemma 2, it follows that $a \equiv b$. Consequently $\pi \equiv \equiv$.

**Corollary 1.** If $S$ is a duo semigroup, then $S/\pi$ is a maximal separative homomorphic image of $S$.

**Corollary 2.** If $S$ is a normal semigroup (i.e. $aS = Sa$ for any $a \in S$), then $S/\pi$ is a maximal separative homomorphic image of $S$.

**Corollary 3.** If $S$ is a quasicommutative semigroup (i.e. for any $a, b \in S$, we have $ab = b'a$ for a positive integer $r$), then $S/\pi$ is a maximal separative homomorphic image of $S$.

**References**


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632