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Czechoslovak Mathematical Journal, Vol. 32 (1982), No. 4, 633–639

Persistent URL: http://dml.cz/dmlcz/101841

Czechoslovak Mathematical Journal, 32 (107) 1982, Praha

ASYMPTOTIC EQUIVALENCE OF DIFFERENTIAL EQUATIONS WITH STEPANOFF-BOUNDED FUNCTIONAL PERTURBATION

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(Received August 31, 1981)

1. INTRODUCTION AND PRELIMINARIES

The purpose of this paper is the study of a generalized asymptotic equivalence between the solutions of the differential equations

$$\text{(I)} \qquad y'(t) = A(t)\, y(t), \quad (t, y) \in R \times R^n$$

and

$$\text{(II)} \qquad x'(t) = A(t)\, x(t) + f(t, T(t, x)), \quad (t, x) \in R \times R^n,$$

where $A(t)$ is an $n \times n$ matrix locally integrable on R, B a given compact subset of R, $T: R \times C[R, R^n] \to C[B, R^n]$ is defined by

$$T(t, x)(\vartheta) = x(\alpha(t, \vartheta)), \quad \vartheta \in B$$

for given $\alpha \in C[R \times B, R]$, and $f: R \times C[B, R^n] \to R^n$ satisfies the Caratheodory conditions, i.e. $f(t, \psi)$ is measurable in t for all $\psi \in C[B, R^n]$ and continuous in ψ for all $t \in R$.

For this problem results are known in which the perturbation term is bounded by a function having zero limit as $|t| \to \infty$ (Hallam [3]). Lovelady [4] relaxed the condition on the asymptotic estimate of the nonlinear perturbation f, at the cost of slightly strengthening the conditions on the linear equation (I).

In the present work, using the basic idea of Lovelady, we employ Stepanoff-like conditions on f and prove the existence of a homeomorphism H between the sets of bounded solutions of (I) and (II). We are going to study the asymptotic relationship between equations (I) and (II), such that to each bounded solution $x(t) = H\, y(t)$ of (II) we have $\lim |y(t) - H\, y(t)| = 0$ as $|t| \to \infty$.

We consider the case in which the linear homogeneous equation (I) is conditionally asymptotically stable.

It is necessary to impose hypotheses upon the linear equation (I) based on the decomposition of R^n into the direct sum

$$R^n = X_0 \oplus X_{-1} \oplus X_1 \oplus X_\infty ,$$

where the subspaces X_i, $i = 0, \pm 1, \infty$, are determined in the folowing manner: denote by $y(t; 0, y_0)$ the solution of (I) starting from y_0 at 0; then $y_0 \in X_0$ if and only if the solution $y(t; 0, y_0)$ is bounded on R; $y_0 \in X_{-1} \oplus X_0$ if and only if the solution $y(t; 0, y_0)$ is bounded on $[0, \infty)$; $y_0 \in X_0 \oplus X_1$ if and only if the solution $y(t; 0, y_0)$ is bounded on $(-\infty, 0]$; X_∞ is the direct complement of $X_0 \oplus X_{-1} \oplus X_1$. We denote by P_i, $i = 0, \pm 1, \infty$, the corresponding projections, i.e.

$$P_i R^n = X_i , \quad i = 0, \pm 1, \infty .$$

If $Y(t)$ is the fundamental matrix solution of (I), then in terms of the above projections, the solution $y(t; 0, y)$ can be written as

$$y(t; 0, y_0) = [\sum_i \Phi_i(t; s)] \, y_0 , \quad i = 0, \pm 1, \infty ,$$

where $\Phi_i(t; t_0) = Y(t) \, P_i Y^{-1}(t_0)$, $i = 0, \pm 1, \infty$.

2. MAIN RESULTS

The following lemma will be used in the sequel. Its proof is quite straightforward (for details cf. [2]).

Lemma 1. *Let C be the Banach space of bounded continuous functions $x = x(t)$ from R to R^n with the norm $\|x\| = \sup_{t \in R} x(t)$. Let $F : C \to C$ be a contraction and U, V nonempty subsets of C such that $(I - F) \, V \subset U$ (I the identity operator). If $H : U \to \to V$ satisfies the relation*

$$H \, y(t) = y(t) + FH \, y(t) , \quad y \in U , \quad t \in R ,$$

then H is a homeomorphism of U onto V.

Theorem 1. *Suppose that equations (I) and (II) satisfy the following hypotheses:*

(i) *There exist supplementary projections P_i, $i = 0, \pm 1, \infty$, and constants q, K ($K > 0$ and $1 < q < \infty$) such that*

$$\sum_{k=t}^{-\infty} \left[\int_{k-1}^{k} |\Phi_{-1}(t; s)|^q \, ds \right]^{1/q} + \sum_{\substack{k=0, \text{if} t \geqq 0 \\ k=-1, \text{if} t < 0}}^{t} \left[\int_{k}^{k+1} |\Phi_0(t; s)|^q \, ds \right]^{1/q} +$$

$$+ \sum_{k=t}^{\infty} \left[\int_{k}^{k+1} |\Phi_1(t; s)|^q \, ds \right]^{1/q} \leqq K ,$$

(ii) *for all* $(t, \psi) \in R \times C[B, R^n]$,

$$P_\infty Y^{-1}(t) f(t, \psi) = 0,$$

(iii) *there exists a function* $\gamma : R \to R^n$ *such that* $[\int_t^{t+1} |\gamma(s)|^p \, ds]^{1/p}$ *exists,*

$$\sup \left[\int_t^{t+1} |\gamma(s)|^p \, ds\right]^{1/p} < K^{-1} \quad \textit{for every} \quad t \in R, \quad \textit{where} \quad p + q = pq,$$

and for every $(t, \psi_1), (t, \psi_2) \in R \times C[B, R^n]$,

$$|f(t, \psi_1) - f(t, \psi_2)| \leqq \gamma(t) \, |\psi_1 - \psi_2|_B,$$

where $|u|_B = \sup_{t \in B} |u(t)|$,

(iv)
$$\sup_{t \in R} \left[\int_t^{t+1} |f(s, 0)|^p \, ds\right]^{1/p} < \infty, \quad p + q = pq.$$

Then there exists a homeomorphism H from the set of bounded solutions of (I) *onto the bounded solutions of* (II), *such that for every* $y = y(t)$ *we have*

(1.1)
$$P_0 H \, y(0) = P_0 \, y(0), \quad P_\infty H \, y(0) = P_\infty \, y(0),$$

$$P_1 H \, y(0) = P_1 \, y(0) - P_1 \int_0^\infty Y^{-1}(s) f(s, T(s, Hy)) \, ds,$$

$$P_{-1} H \, y(0) = P_{-1} \, y(0) + P_{-1} \int_{-\infty}^0 Y^{-1}(s) f(s, T(s, Hy)) \, ds.$$

Proof. We define an operator F on C by the equation

$$F \, x(t) = \int_{-\infty}^t \Phi_{-1}(t; s) f(s, T(s, x)) \, ds + \int_0^t \Phi_0(t; s) f(s, T(s, x)) \, ds -$$

$$- \int_t^\infty \Phi_1(t; s) f(s, T(s, x)) \, ds, \quad x \in C, \quad t \in R.$$

By condition (i), Hölder and Minkowski inequalities and the fact that condition (iii) implies $|f(t, T(t, x))| \leqq \gamma(t) \, \|x\| + |f(t, 0)|$ it follows that for $t \geqq 0$,

$|F \, x(t)| \leqq$

$$\leqq \sum_{k=t}^{-\infty} \left(\left[\int_{k-1}^k (\gamma(s) \| x\|)^p \, ds\right]^{1/p} + \left[\int_{k-1}^k |f(s, 0)^p \, ds\right]^{1/p}\right) \left[\int_{k-1}^k |\Phi_{-1}(t; s)|^q \, ds\right]^{1/q} +$$

$$+ \sum_{k=0}^t \left(\left[\int_k^{k+1} (\gamma(s) \, \|x\|)^p \, ds\right]^{1/p} + \left[\int_k^{k+1} |f(s, 0)|^p \, ds\right]^{1/p}\right) \left[\int_k^{k+1} |\Phi_0(t; s)|^q \, ds\right]^{1/q} +$$

$$+\sum_{k=t}^{\infty}\left(\left[\int_k^{k+1}(\gamma(s)\,\|x\|)^p\,\mathrm{d}s\right]^{1/p}+\left[\int_k^{k+1}|f(s,0)|^p\,\mathrm{d}s\right]^{1/p}\right)\left[\int_k^{k+1}|\Phi_1(t;s)|^q\,\mathrm{d}s\right]^{1/q}\leqq$$

$$\leqq\sup_{-\infty\leqq k\leqq t}\left(\left[\int_{k-1}^{k}(\gamma(s)\,\|x\|)^p\,\mathrm{d}s\right]^{1/p}+\right.$$

$$\left.+\left[\int_{k-1}^{k}|f(s,0)|^p\,\mathrm{d}s\right]^{1/p}\right)\sum_{k=t}^{-\infty}\left[\int_{k-1}^{k}|\Phi_{-1}(t;s)|^q\,\mathrm{d}s\right]^{1/q}+$$

$$+\sup_{0\leqq k\leqq t}\left(\left[\int_k^{k+1}(\gamma(s)\,\|x\|)^p\,\mathrm{d}s\right]^{1/p}+\right.$$

$$\left.+\left[\int_k^{k+1}|f(s,0)|^p\,\mathrm{d}s\right]^{1/p}\right)\sum_{k=0}^{t}\left[\int_k^{k+1}|\Phi_0(t;s)|^q\,\mathrm{d}s\right]^{1/q}+$$

$$+\sup_{t\leqq k\leqq\infty}\left(\left[\int_k^{k+1}(\gamma(s)\,\|x\|)^p\,\mathrm{d}s\right]^{1/p}+\right.$$

$$\left.+\left[\int_k^{k+1}|f(s,0)^p\,\mathrm{d}s\right]^{1/p}\right)\sum_{k=t}^{\infty}\left[\int_k^{k+1}|\Phi_1(t;s)|^q\,\mathrm{d}s\right]^{1/q}\leqq$$

$$\leqq\left(\|x\|\sup_{k\in R}\left[\int_k^{k+1}|\gamma(s)|^p\,\mathrm{d}s\right]^{1/p}+\sup_{k\in R}\left[\int_k^{k+1}|f(s,0)|^p\,\mathrm{d}s\right]^{1/p}\right)K\,.$$

Thus $Fx\in C$.

For every fixed bounded solution y of (I) we define an operator $S_y : C\to C$, by the relation

$$S_y\,x(t)=y(t)+F\,x(t)\,,\quad x\in C,\quad t\in R\,.$$

We will demonstrate that S_y has a unique fixed point in C by using the Banach contraction principle.

From the definitions we have that $|T(t,x)|_B\leqq\|x\|$. So, for $x_1,x_2\in C$, and using Hölder inequality we have for $t\geqq 0$

$$\|Fx_1-Fx_2\|\leqq\|x_1-x_2\|\,K\sup_{k\in R}\left[\int_k^{k+1}|\gamma(s)|^p\,\mathrm{d}s\right]^{1/p}.$$

The proof for $t<0$ is similar. This implies that F is a contraction and so S_y is a contraction, too.

An easy computation shows that the fixed point $x(t)=S_y\,x(t)$, $t\in R$, is a solution of (II).

Let C_I, C_{II} be the spaces of bounded solutions of equations (I) and (II), respectively. We define the mapping $H : C_I\to C_{II}$ in the following way: for every $y\in C_I$, Hy will be the fixed point of the contraction S_y. Thus, for $t \in R$,

$$H\, y(t) = S_y H\, y(t)\,.$$

According to Lemma 1, setting $U = C_I$ and $V = C_{II}$, H is a homeomorphism from C_I to C_{II} and the inverse mapping is

$$H^{-1}\, x(t) = x(t) - F\, x(t)\,, \quad x \in C_{II}\,, \quad t \in R\,.$$

If we put $t = 0$, we have

$$H\, y(0) = y(0) + P_{-1} \int_{-\infty}^{0} Y^{-1}(s) f(s,\, T(s,\, Hy))\, \mathrm{d}s - P_1 \int_{0}^{\infty} Y^{-1}(s) f(s,\, T(s,\, Hy))\, \mathrm{d}s$$

and so we obtain relations (1.1).

Theorem 2. *Suppose that equations.* (I) *and* (II) *satisfy conditions* (i), (ii), (iii) *and* (iv) *of Theorem* 1. *Moreover, suppose that*

$$\text{(v)} \qquad \lim_{|t| \to \infty} \left[\int_t^{t+1} |\gamma(s)|^p\, \mathrm{d}s \right]^{1/p} = 0$$

and

$$\text{(vi)} \qquad \lim_{|t| \to \infty} \left[\int_t^{t+1} |f(s,\, 0)|^p\, \mathrm{d}s \right]^{1/p} = 0\,.$$

Then, under these hypotheses, for every $y \in C_I$ *the relation*

$$\lim_{|t| \to \infty} |y(t) - H\, y(t)| = 0$$

is satisfied.

Proof. According to conditions (v) and (vi) for a given $\varepsilon > 0$, we can choose $t_2 > 0$ such that for $|k| \geqq t_2$, the following relations hold:

$$\|Hy\| \left[\int_{k-1}^{k} |\gamma(s)|^p\, \mathrm{d}s \right]^{1/p} < \frac{\varepsilon}{3K}\,, \quad \left[\int_{k-1}^{k} |f(s,\, 0)|^p\, \mathrm{d}s \right]^{1/p} < \frac{\varepsilon}{3K}$$

and

$$\|Hy\| \left[\int_{k}^{k+1} |\gamma(s)|^p\, \mathrm{d}s \right]^{1/p} < \frac{\varepsilon}{3K}\,, \quad \left[\int_{k}^{k+1} |f(s,\, 0)|^p\, \mathrm{d}s \right]^{1/p} < \frac{\varepsilon}{3K}\,.$$

Hypothesis (i) of Theorem 1 implies that

$$\left[\int_{-\infty}^{t} |\Phi_{-1}(t;\, s)|^q\, \mathrm{d}s \right]^{1/q} + \left[\left| \int_{0}^{t} |\Phi_0(t;\, s)|^q\, \mathrm{d}s \right| \right]^{1/q} + \left[\int_{t}^{\infty} |\Phi_1(t;\, s)|^q\, \mathrm{d}s \right]^{1/q} \leqq K\,, \quad t \in R\,,$$

from which applying Lemma 2 (i) of [3] we obtain

$$\lim_{t \to \infty} |Y(t) P_i| = 0, \quad i = -1, 0.$$

Hence we can choose $t_3 \geqq t_2$, such that for $t \geqq t_3$ we have

$$|Y(t) P_i| \int_{-t_2}^{t_2} |Y^{-1}(s) f(s, T(s, Hy))| \,ds < \frac{\varepsilon}{6}, \quad i = -1, 0,$$

so

$$|y(t) - H y(t)| \leqq$$

$$\leqq \sum_{k=-t_2}^{-\infty} \left[\int_{k-1}^{k} [\|Hy\| \gamma(s) + |f(s,0)|]^p \,ds\right]^{1/p} \left[\int_{k-1}^{k} |\Phi_{-1}(t; s)|^q \,ds\right]^{1/q} +$$

$$+ |Y(t) P_{-1}| \int_{-t_2}^{t_2} |Y^{-1}(s) f(s, T(s, Hy))| \,ds +$$

$$+ \sum_{k=t}^{t_2+1} \left[\int_{k-1}^{k} [\|Hy\| \gamma(s) + |f(s,0)|]^p \,ds\right]^{1/p} \left[\int_{k-1}^{k} |\Phi_{-1}(t; s)|^q \,ds\right]^{1/q} +$$

$$+ |Y(t) P_0| \int_{0}^{t_2} |Y^{-1}(s) f(s, T(s, Hy))| \,ds +$$

$$+ \sum_{k=t_2}^{t} \left[\int_{k}^{k+1} [\|Hy\| \gamma(s) + |f(s,0)|]^p \,ds\right]^{1/p} \left[\int_{k}^{k+1} |\Phi_0(t; s)|^q \,ds\right]^{1/q} +$$

$$+ \sum_{k=t}^{\infty} \left[\int_{k}^{k+1} [\|Hy\| \gamma(s) + |f(s,0)|]^p \,ds\right]^{1/p} \left[\int_{k}^{k+1} |\Phi_1(t; s)|^q \,ds\right]^{1/q} \leqq$$

$$\leqq \left[\!\left[\|Hy\| \sup_{k \leqq -t_2} \left[\int_{k-1}^{k} |\gamma(s)|^p \,ds\right]^{1/p} + \sup_{k \leqq t_2} \left[\int_{k}^{k+1} |\gamma(s)|^p \,ds|^{1/p}\right] + \right.\right.$$

$$\left.\left. + \sup_{k \geqq -t_2} \left[\int_{k-1}^{k} |f(s,0)|^p \,ds\right]^{1/p} + \sup_{k \geqq t_2} \left[\int_{k}^{k+1} f(s,0)|^p \,ds\right]^{1/p}\right]\!\right] K + \frac{\varepsilon}{3} \leqq$$

$$\leqq \left[\frac{\varepsilon}{3K} + \frac{\varepsilon}{3K}\right] K + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore

$$\lim_{t \to +\infty} |y(t) - H y(t)| = 0.$$

In a similar way, applying Lemma 2 (ii) of [3], we get $\lim |y(t) - H y(t)| = 0$, which completes the proof.

We remark that the present results extend those of [1] as we prove here the existence of a homeomorphism through the contraction mapping principle. In [1] the basic tool was Schauder's fixed point theorem.

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