

Moses A. Boudourides; Dimitris N. Georgiou

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ASYMPTOTIC EQUIVALENCE OF DIFFERENTIAL EQUATIONS  
WITH STEPANOFF-BOUNDED FUNCTIONAL PERTURBATION

M. BOUDOURIDES, Paris, D. GEORGIU, Xanthi

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1. INTRODUCTION AND PRELIMINARIES

The purpose of this paper is the study of a generalized asymptotic equivalence between the solutions of the differential equations

$$(I) \quad y'(t) = A(t) y(t), \quad (t, y) \in R \times R^n$$

and

$$(II) \quad x'(t) = A(t) x(t) + f(t, T(t, x)), \quad (t, x) \in R \times R^n,$$

where  $A(t)$  is an  $n \times n$  matrix locally integrable on  $R$ ,  $B$  a given compact subset of  $R$ ,  $T: R \times C[R, R^n] \rightarrow C[B, R^n]$  is defined by

$$T(t, x)(\vartheta) = x(\alpha(t, \vartheta)), \quad \vartheta \in B$$

for given  $\alpha \in C[R \times B, R]$ , and  $f: R \times C[B, R^n] \rightarrow R^n$  satisfies the Caratheodory conditions, i.e.  $f(t, \psi)$  is measurable in  $t$  for all  $\psi \in C[B, R^n]$  and continuous in  $\psi$  for all  $t \in R$ .

For this problem results are known in which the perturbation term is bounded by a function having zero limit as  $|t| \rightarrow \infty$  (Hallam [3]). Lovelady [4] relaxed the condition on the asymptotic estimate of the nonlinear perturbation  $f$ , at the cost of slightly strengthening the conditions on the linear equation (I).

In the present work, using the basic idea of Lovelady, we employ Stepanoff-like conditions on  $f$  and prove the existence of a homeomorphism  $H$  between the sets of bounded solutions of (I) and (II). We are going to study the asymptotic relationship between equations (I) and (II), such that to each bounded solution  $x(t) = H y(t)$  of (II) we have  $\lim |y(t) - H y(t)| = 0$  as  $|t| \rightarrow \infty$ .

We consider the case in which the linear homogeneous equation (I) is conditionally asymptotically stable.

It is necessary to impose hypotheses upon the linear equation (I) based on the de-

composition of  $R^n$  into the direct sum

$$R^n = X_0 \oplus X_{-1} \oplus X_1 \oplus X_\infty,$$

where the subspaces  $X_i$ ,  $i = 0, \pm 1, \infty$ , are determined in the following manner: denote by  $y(t; 0, y_0)$  the solution of (I) starting from  $y_0$  at 0; then  $y_0 \in X_0$  if and only if the solution  $y(t; 0, y_0)$  is bounded on  $R$ ;  $y_0 \in X_{-1} \oplus X_0$  if and only if the solution  $y(t; 0, y_0)$  is bounded on  $[0, \infty)$ ;  $y_0 \in X_0 \oplus X_1$  if and only if the solution  $y(t; 0, y_0)$  is bounded on  $(-\infty, 0]$ ;  $X_\infty$  is the direct complement of  $X_0 \oplus X_{-1} \oplus X_1$ . We denote by  $P_i$ ,  $i = 0, \pm 1, \infty$ , the corresponding projections, i.e.

$$P_i R^n = X_i, \quad i = 0, \pm 1, \infty.$$

If  $Y(t)$  is the fundamental matrix solution of (I), then in terms of the above projections, the solution  $y(t; 0, y)$  can be written as

$$y(t; 0, y_0) = \left[ \sum_i \Phi_i(t; s) \right] y_0, \quad i = 0, \pm 1, \infty,$$

where  $\Phi_i(t; t_0) = Y(t) P_i Y^{-1}(t_0)$ ,  $i = 0, \pm 1, \infty$ .

## 2. MAIN RESULTS

The following lemma will be used in the sequel. Its proof is quite straightforward (for details cf. [2]).

**Lemma 1.** *Let  $C$  be the Banach space of bounded continuous functions  $x = x(t)$  from  $R$  to  $R^n$  with the norm  $\|x\| = \sup_{t \in R} \|x(t)\|$ . Let  $F : C \rightarrow C$  be a contraction and  $U, V$  nonempty subsets of  $C$  such that  $(I - F)V \subset U$  ( $I$  the identity operator). If  $H : U \rightarrow V$  satisfies the relation*

$$H y(t) = y(t) + FH y(t), \quad y \in U, \quad t \in R,$$

then  $H$  is a homeomorphism of  $U$  onto  $V$ .

**Theorem 1.** *Suppose that equations (I) and (II) satisfy the following hypotheses:*

- (i) *There exist supplementary projections  $P_i$ ,  $i = 0, \pm 1, \infty$ , and constants  $q, K$  ( $K > 0$  and  $1 < q < \infty$ ) such that*

$$\begin{aligned} & \sum_{k=t}^{-\infty} \left[ \int_{k-1}^k |\Phi_{-1}(t; s)|^q ds \right]^{1/q} + \sum_{\substack{k=0, \text{ if } t \geq 0 \\ k=-1, \text{ if } t < 0}}^t \left[ \int_k^{k+1} |\Phi_0(t; s)|^q ds \right]^{1/q} + \\ & + \sum_{k=t}^{\infty} \left[ \int_k^{k+1} |\Phi_1(t; s)|^q ds \right]^{1/q} \leq K, \end{aligned}$$

(ii) for all  $(t, \psi) \in R \times C[B, R^n]$ ,

$$P_\infty Y^{-1}(t)f(t, \psi) = 0,$$

(iii) there exists a function  $\gamma : R \rightarrow R^n$  such that  $[\int_t^{t+1} |\gamma(s)|^p ds]^{1/p}$  exists,

$$\sup \left[ \int_t^{t+1} |\gamma(s)|^p ds \right]^{1/p} < K^{-1} \text{ for every } t \in R, \text{ where } p + q = pq,$$

and for every  $(t, \psi_1), (t, \psi_2) \in R \times C[B, R^n]$ ,

$$|f(t, \psi_1) - f(t, \psi_2)| \leq \gamma(t) |\psi_1 - \psi_2|_B,$$

where  $|u|_B = \sup_{t \in B} |u(t)|$ ,

$$(iv) \quad \sup_{t \in R} \left[ \int_t^{t+1} |f(s, 0)|^p ds \right]^{1/p} < \infty, \quad p + q = pq.$$

Then there exists a homeomorphism  $H$  from the set of bounded solutions of (I) onto the bounded solutions of (II), such that for every  $y = y(t)$  we have

$$(1.1) \quad P_0 H y(0) = P_0 y(0), \quad P_\infty H y(0) = P_\infty y(0),$$

$$P_1 H y(0) = P_1 y(0) - P_1 \int_0^\infty Y^{-1}(s) f(s, T(s, Hy)) ds,$$

$$P_{-1} H y(0) = P_{-1} y(0) + P_{-1} \int_{-\infty}^0 Y^{-1}(s) f(s, T(s, Hy)) ds.$$

Proof. We define an operator  $F$  on  $C$  by the equation

$$F x(t) = \int_{-\infty}^t \Phi_{-1}(t; s) f(s, T(s, x)) ds + \int_0^t \Phi_0(t; s) f(s, T(s, x)) ds - \int_t^\infty \Phi_1(t; s) f(s, T(s, x)) ds, \quad \lambda \in C, \quad t \in R.$$

By condition (i), Hölder and Minkowski inequalities and the fact that condition (iii) implies  $|f(t, T(t, x))| \leq \gamma(t) \|x\| + |f(t, 0)|$  it follows that for  $t \geq 0$ ,

$$\begin{aligned} |F x(t)| &\leq \\ &\leq \sum_{k=t}^{-\infty} \left( \left[ \int_{k-1}^k (\gamma(s) \|x\|)^p ds \right]^{1/p} + \left[ \int_{k-1}^k |f(s, 0)|^p ds \right]^{1/p} \right) \left[ \int_{k-1}^k |\Phi_{-1}(t; s)|^q ds \right]^{1/q} + \\ &+ \sum_{k=0}^t \left( \left[ \int_k^{k+1} (\gamma(s) \|x\|)^p ds \right]^{1/p} + \left[ \int_k^{k+1} |f(s, 0)|^p ds \right]^{1/p} \right) \left[ \int_k^{k+1} |\Phi_0(t; s)|^q ds \right]^{1/q} + \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=t}^{\infty} \left( \left[ \int_k^{k+1} (\gamma(s) \|x\|)^p ds \right]^{1/p} + \left[ \int_k^{k+1} |f(s, 0)|^p ds \right]^{1/p} \right) \left[ \int_k^{k+1} |\Phi_1(t; s)|^q ds \right]^{1/q} \leq \\
& \leq \sup_{-\infty \leq k \leq t} \left( \left[ \int_{k-1}^k (\gamma(s) \|x\|)^p ds \right]^{1/p} + \right. \\
& + \left. \left[ \int_{k-1}^k |f(s, 0)|^p ds \right]^{1/p} \right) \sum_{k=t}^{-\infty} \left[ \int_{k-1}^k |\Phi_{-1}(t; s)|^q ds \right]^{1/q} + \\
& + \sup_{0 \leq k \leq t} \left( \left[ \int_k^{k+1} (\gamma(s) \|x\|)^p ds \right]^{1/p} + \right. \\
& + \left. \left[ \int_k^{k+1} |f(s, 0)|^p ds \right]^{1/p} \right) \sum_{k=0}^t \left[ \int_k^{k+1} |\Phi_0(t; s)|^q ds \right]^{1/q} + \\
& + \sup_{t \leq k \leq \infty} \left( \left[ \int_k^{k+1} (\gamma(s) \|x\|)^p ds \right]^{1/p} + \right. \\
& + \left. \left[ \int_k^{k+1} |f(s, 0)|^p ds \right]^{1/p} \right) \sum_{k=t}^{\infty} \left[ \int_k^{k+1} |\Phi_1(t; s)|^q ds \right]^{1/q} \leq \\
& \leq \left( \|x\| \sup_{k \in \mathbb{R}} \left[ \int_k^{k+1} |\gamma(s)|^p ds \right]^{1/p} + \sup_{k \in \mathbb{R}} \left[ \int_k^{k+1} |f(s, 0)|^p ds \right]^{1/p} \right) K.
\end{aligned}$$

Thus  $Fx \in C$ .

For every fixed bounded solution  $y$  of (I) we define an operator  $S_y : C \rightarrow C$ , by the relation

$$S_y x(t) = y(t) + F x(t), \quad x \in C, \quad t \in \mathbb{R}.$$

We will demonstrate that  $S_y$  has a unique fixed point in  $C$  by using the Banach contraction principle.

From the definitions we have that  $|T(t, x)|_B \leq \|x\|$ . So, for  $x_1, x_2 \in C$ , and using Hölder inequality we have for  $t \geq 0$

$$\|Fx_1 - Fx_2\| \leq \|x_1 - x_2\| K \sup_{k \in \mathbb{R}} \left[ \int_k^{k+1} |\gamma(s)|^p ds \right]^{1/p}.$$

The proof for  $t < 0$  is similar. This implies that  $F$  is a contraction and so  $S_y$  is a contraction, too.

An easy computation shows that the fixed point  $x(t) = S_y x(t)$ ,  $t \in \mathbb{R}$ , is a solution of (II).

Let  $C_I, C_{II}$  be the spaces of bounded solutions of equations (I) and (II), respectively. We define the mapping  $H : C_I \rightarrow C_{II}$  in the following way: for every  $y \in C_I$ ,  $H y$  will

be the fixed point of the contraction  $S_y$ . Thus, for  $t \in R$ ,

$$H y(t) = S_y H y(t).$$

According to Lemma 1, setting  $U = C_I$  and  $V = C_{II}$ ,  $H$  is a homeomorphism from  $C_I$  to  $C_{II}$  and the inverse mapping is

$$H^{-1} x(t) = x(t) - F x(t), \quad x \in C_{II}, \quad t \in R.$$

If we put  $t = 0$ , we have

$$H y(0) = y(0) + P_{-1} \int_{-\infty}^0 Y^{-1}(s) f(s, T(s, Hy)) ds - P_1 \int_0^{\infty} Y^{-1}(s) f(s, T(s, Hy)) ds$$

and so we obtain relations (1.1).

**Theorem 2.** Suppose that equations (I) and (II) satisfy conditions (i), (ii), (iii) and (iv) of Theorem 1. Moreover, suppose that

$$(v) \quad \lim_{|t| \rightarrow \infty} \left[ \int_t^{t+1} |\gamma(s)|^p ds \right]^{1/p} = 0$$

and

$$(vi) \quad \lim_{|t| \rightarrow \infty} \left[ \int_t^{t+1} |f(s, 0)|^p ds \right]^{1/p} = 0.$$

Then, under these hypotheses, for every  $y \in C_I$  the relation

$$\lim_{|t| \rightarrow \infty} |y(t) - H y(t)| = 0$$

is satisfied.

*Proof.* According to conditions (v) and (vi) for a given  $\varepsilon > 0$ , we can choose  $t_2 > 0$  such that for  $|k| \geq t_2$ , the following relations hold:

$$\|Hy\| \left[ \int_{k-1}^k |\gamma(s)|^p ds \right]^{1/p} < \frac{\varepsilon}{3K}, \quad \left[ \int_{k-1}^k |f(s, 0)|^p ds \right]^{1/p} < \frac{\varepsilon}{3K}$$

and

$$\|Hy\| \left[ \int_k^{k+1} |\gamma(s)|^p ds \right]^{1/p} < \frac{\varepsilon}{3K}, \quad \left[ \int_k^{k+1} |f(s, 0)|^p ds \right]^{1/p} < \frac{\varepsilon}{3K}.$$

Hypothesis (i) of Theorem 1 implies that

$$\left[ \int_{-\infty}^t |\Phi_{-1}(t; s)|^q ds \right]^{1/q} + \left[ \int_0^t |\Phi_0(t; s)|^q ds \right]^{1/q} + \left[ \int_t^{\infty} |\Phi_1(t; s)|^q ds \right]^{1/q} \leq K, \quad t \in R,$$

from which applying Lemma 2 (i) of [3] we obtain

$$\lim_{t \rightarrow \infty} |Y(t) P_i| = 0, \quad i = -1, 0.$$

Hence we can choose  $t_3 \geq t_2$ , such that for  $t \geq t_3$  we have

$$|Y(t) P_i| \int_{-t_2}^{t_2} |Y^{-1}(s) f(s, T(s, Hy))| ds < \frac{\varepsilon}{6}, \quad i = -1, 0,$$

so

$$\begin{aligned} & |y(t) - Hy(t)| \leq \\ & \leq \sum_{k=-t_2}^{-\infty} \left[ \int_{k-1}^k [\|Hy\| \gamma(s) + |f(s, 0)|]^p ds \right]^{1/p} \left[ \int_{k-1}^k |\Phi_{-1}(t; s)|^q ds \right]^{1/q} + \\ & \quad + |Y(t) P_{-1}| \int_{-t_2}^{t_2} |Y^{-1}(s) f(s, T(s, Hy))| ds + \\ & + \sum_{k=t}^{t_2+1} \left[ \int_{k-1}^k [\|Hy\| \gamma(s) + |f(s, 0)|]^p ds \right]^{1/p} \left[ \int_{k-1}^k |\Phi_{-1}(t; s)|^q ds \right]^{1/q} + \\ & \quad + |Y(t) P_0| \int_0^{t_2} |Y^{-1}(s) f(s, T(s, Hy))| ds + \\ & + \sum_{k=t_2}^t \left[ \int_k^{k+1} [\|Hy\| \gamma(s) + |f(s, 0)|]^p ds \right]^{1/p} \left[ \int_k^{k+1} |\Phi_0(t; s)|^q ds \right]^{1/q} + \\ & + \sum_{k=t}^{\infty} \left[ \int_k^{k+1} [\|Hy\| \gamma(s) + |f(s, 0)|]^p ds \right]^{1/p} \left[ \int_k^{k+1} |\Phi_1(t; s)|^q ds \right]^{1/q} \leq \\ & \leq \left[ \|Hy\| \sup_{k \leq -t_2} \left[ \int_{k-1}^k |\gamma(s)|^p ds \right]^{1/p} + \sup_{k \leq t_2} \left[ \int_k^{k+1} |\gamma(s)|^p ds \right]^{1/p} \right] + \\ & + \sup_{k \geq -t_2} \left[ \int_{k-1}^k |f(s, 0)|^p ds \right]^{1/p} + \sup_{k \geq t_2} \left[ \int_k^{k+1} |f(s, 0)|^p ds \right]^{1/p} \Big] K + \frac{\varepsilon}{3} \leq \\ & \leq \left[ \frac{\varepsilon}{3K} + \frac{\varepsilon}{3K} \right] K + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore

$$\lim_{t \rightarrow +\infty} |y(t) - Hy(t)| = 0.$$

In a similar way, applying Lemma 2 (ii) of [3], we get  $\lim |y(t) - Hy(t)| = 0$ , which completes the proof.

We remark that the present results extend those of [1] as we prove here the existence of a homeomorphism through the contraction mapping principle. In [1] the basic tool was Schauder's fixed point theorem.

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*Authors' addresses*: M. Boudourides, Analyse Numérique, Université Pierre et Marie Curie, 75230 Paris, France; D. Georgiou, Department of Mathematics, Democritus University of Thrace, Xanthi, Greece.