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THE CATEGORY OF PAWLAK MACHINES

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1. INTRODUCTION

Iterative systems, first defined by Pawlak [1], are very simplified models of digital computers. Though it is not possible to express some properties of mathematical machines (e.g. the concept of program), on the other hand the simplicity of the accepted model enables advanced algebraic studies.

This paper is concerned with the study of the category whose objects are Pawlak machines and where morphisms are simulations. By a Pawlak machine, we understand a nonempty set with one partial unary operation. The simulation is a mapping which has the following intuitive interpretation: the machine B simulates the activity of the machine A if one step in A corresponds to one or more steps in B and halting of A forces B to halt as well.

Two problems are investigated: 1) find necessary and sufficient conditions for the existence of simulations for given two Pawlak machines; 2) describe the category of Pawlak machines.

2. BASIC NOTIONS

We denote by Ord the class of all ordinals and by N the set of all finite ordinals. If $\alpha \in \mathrm{Ord}$ then we put $W(\alpha) = \{\beta \in \mathrm{Ord};\ \beta < \alpha\}$.

Let $\mathscr{A}$ be a category. Then we denote by ob $\mathscr{A}$ the class of objects of $\mathscr{A}$ and by $(P, Q)_{\mathscr{A}}$ the set of all morphisms of P into Q for any $P, Q \in \mathrm{ob}\ \mathscr{A}$.

If $\mathscr{A}$ is a category such that, for any $P, Q \in \mathrm{ob}\ \mathscr{A}$, card $(P, Q)_{\mathscr{A}} \leqq 1$ then $\mathscr{A}$ is called a *thin category*. Moreover, if $(P, Q)_{\mathscr{A}} \neq \emptyset$, $(Q, P)_{\mathscr{A}} \neq \emptyset$ implies $P = Q$, then $\mathscr{A}$ is called an *ordered class*. If $\mathscr{A}$ is a thin category (an ordered class respectively) then we put $P \prod_{\mathscr{A}} Q$ ($P \leqq_{\mathscr{A}} Q$ respectively) iff $(P, Q)_{\mathscr{A}} \neq \emptyset$.

An ordered class $\mathscr{A}$ is called a *chain* if $P \leqq_{\mathscr{A}} Q$ or $Q \leqq_{\mathscr{A}} P$ for any $P, Q \in \mathrm{ob}\ \mathscr{A}$.

A category $\mathscr{A}$ is called an *antichain* if $(P, Q)_{\mathscr{A}} = \emptyset$ for any different $P, Q \in \mathrm{ob}\ \mathscr{A}$.

Let $\{\mathscr{A}_G;\ G \in \mathscr{G}\}$ be a system of mutually disjoint thin categories indexed by elements of an ordered class $\mathscr{G}$. The lexicographic sum $\sum^{l}_{G \in \mathscr{G}} \mathscr{A}_G$ of the given system is the thin category $\mathscr{A}$, whose class of objects is $\bigcup_{G \in \mathscr{G}} \text{ob}\, \mathscr{A}_G$ and morphisms are defined as follows: Let $K \in \text{ob}\, \mathscr{A}_{G_1}$, $L \in \text{ob}\, \mathscr{A}_{G_2}$, $G_1, G_2 \in \mathscr{G}$. We put $(K, L)_{\mathscr{A}} \neq \emptyset$, if some of the following conditions is valid: (a) $G_1 < G_2$, (b) $G_1 = G_2$ and $K \prod \mathscr{A}_{G_1} L$.

If $\mathscr{G} = \{1, 2, \ldots, n\}$ is an antichain or a chain with the natural order resp. then we put $\sum^{l}_{G \in \mathscr{G}} \mathscr{A}_G = \mathscr{A}_1 + \mathscr{A}_2 + \ldots + \mathscr{A}_n$, $\sum^{l}_{G \in \mathscr{G}} \mathscr{A}_G = \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \ldots \oplus \mathscr{A}_n$ respectively.

Let $\mathscr{A}, \mathscr{B}$ be ordered classes. By a cardinal product we understand the ordered class $\mathscr{C} = \mathscr{A} . \mathscr{B}$ such that $\text{ob}\, \mathscr{C} = \text{ob}\, \mathscr{A} \times \text{ob}\, \mathscr{B}$ and, for any $(P_1, Q_1), (P_2, Q_2) \in \text{ob}\, \mathscr{C}$, $(P_1, Q_1) \leqq_{\mathscr{C}} (P_2, Q_2)$ iff $P_1 \leqq_{\mathscr{A}} P_2$, $Q_1 \leqq_{\mathscr{B}} Q_2$. Compare [5].

Let $\mathscr{A}$ be a category. Then it is called a *category with non-empty homs*, if, for each $P, Q \in \text{ob}\, \mathscr{A}$, there holds $(P, Q)_{\mathscr{A}} \neq \emptyset$. If $\mathscr{A}$ is a category, then a thin category $\mathscr{A}(b)$ such that $\text{ob}\, \mathscr{A}(b) = \text{ob}\, \mathscr{A}$ and $P \prod_{\mathscr{A}(b)} Q$ iff $(P, Q)_{\mathscr{A}} \neq \emptyset$ for each $P. Q \in \text{ob}\, \mathscr{A}$ is called a *basic category* for $\mathscr{A}$. A basic category $\mathscr{A}(b)$ of $\mathscr{A}$ is a thin category with the same objects and the same existence of morphisms.

Definition. Let A be a nonempty set, f a partial mapping of A into itself. Then the ordered pair $\mathsf{A} = (A, f)$ is called a *Pawlak machine*. Let $\mathsf{A} = (A, f)$ be a machine.

We put a) $DA = A - \text{dom}\, f$, b) $f^0 = \text{id}_A$. Let $n \geqq 0$ be an integer and suppose that the partial mapping f^n of A into itself has been defined. If $x \in A$ is such that $f^n(x)$ is defined and $f^n(x) \in \text{dom}\, f$, then we put $f^{n+1}(x) = f(f^n(x))$, By induction we define f^m for any nonnegative integer m.

Definition. Let $\mathsf{A} = (A, f)$, $\mathsf{B} = (B, g)$ be machines, $\mu : A \to B$ a maping. It is said to be a *simulation of* A *into* B if the following conditions are satisfied:

a) For any $x \in \text{dom}\, f$, there exists an integer $k(x) \geqq 1$ such that $\mu(x) \in \text{dom}\, g^{k(x)}$ and $\mu(f(x)) = g^{k(x)}(\mu(x))$.

b) $\mu(DA) \subseteq DB$.

3. SIMULATIONS OF CONNECTED MACHINES

Let $\mathsf{A} = (A, f)$ be a machine. We put $\varrho A = \{(x, y) \in A \times A;$ there exist $m \geqq 0$, $n \geqq 0$ such that $x \in \text{dom}\, f^m$, $y \in \text{dom}\, f^n$, $f^m(x) = f^n(y)\}$.

Clearly, ϱA is an equivalence on A and its blocks are called *components* of A. A machine is said to be *connected* (abbreviation c-machine) if it has exactly one component.

In this paragraph let A be a c-machine. Then, clearly, DA contains at most one element. If $DA \neq \emptyset$ we denote by dA its only element. Further we define $ZA = \{x \in A;$ there exists $n(x) > 0$ such that $x \in \text{dom}\, f^{n(x)}$ and $f^{n(x)}(x) = x\}$, $RA = \text{card}\, ZA$,

$$KA = \{x \in A - ZA; \text{ there exists a sequence } (x_i)_{i \in N} \text{ such that}$$
$$x_0 = x \text{ and } f(x_{i+1}) = x_i \text{ for any } i \in N\}\ .$$

We put

$$A^0 = \{x \in A;\ f^{-1}(x) = \emptyset\}\,,$$

$$A^\alpha = \{x \in A - \bigcup_{\lambda<\alpha} A^\lambda;\ f^{-1}(x) \subseteq \bigcup_{\lambda<\alpha} A^\lambda\} \quad \text{for any ordinal } \alpha > 0\,;$$

$$\vartheta A = \min\{\lambda \in \mathrm{Ord};\ A^\lambda = \emptyset\}\,.$$

Let $\infty_1, \infty_2 \notin \mathrm{Ord}$ and suppose that $\alpha < \infty_1 < \infty_2$ for each $\alpha \in \mathrm{Ord}$. We define $A^{\infty_1} = KA$, $A^{\infty_2} = ZA$, $SA(x) = \varkappa$ if $x \in A^\varkappa$ for some $\varkappa \in W(\vartheta A) \cup \{\infty_1, \infty_2\}$.

Lemma 1. (i) *If $DA \neq \emptyset$, $A^{\infty_1} \neq \emptyset$, then $SA(dA) = \infty_1$.*
(ii) *If $DA \neq \emptyset$, $A^{\infty_1} = \emptyset$, then ϑA is an isolated ordinal and $SA(dA) = \vartheta A - 1$.*
(iii) *If $DA = \emptyset$, $A^{\infty_1} = \emptyset$, $A^{\infty_2} = \emptyset$ then ϑA is a limit ordinal cofinal with ω_0.*

For the proof see [3], [4]. □

We put

$$0^i = \{\lambda \in \mathrm{Ord};\ \lambda > 0,\ \lambda \text{ an isolated ordinal}\}\,,$$

$$0^l = \{\lambda \in \mathrm{Ord};\ \lambda \text{ a limit ordinal cofinal with } \omega_0\}\,.$$

Let $d_1, d_2, d_3 \notin \mathrm{Ord}$ and suppose that $\alpha < d_1$ for any $\alpha \in 0^i$, $\beta < d_2 < d_3$ for any $\beta \in 0^l$.

Definition. Let A be a c-machine. Then we put

$$\chi_0 A = \begin{cases} d_3 & \text{if } RA \neq 0\,, \\ d_2 & \text{if } RA = 0\,,\quad KA \neq \emptyset\,,\quad DA = \emptyset\,, \\ d_1 & \text{if } RA = 0\,,\quad KA \neq \emptyset\,,\quad DA \neq \emptyset\,, \\ \vartheta A & \text{if } RA = 0\,,\quad KA = \emptyset\,. \end{cases}$$

Definition. Let A, B be c-machines. We put $(A, B) \in sAD$ if
(a) card DA = card DB,
(b) $\chi_0 A \leqq \chi_0 B$.

Lemma 2. *Let A, B be c-machines, $\mu : A \to B$ a simulation.*
(i) *For any $n > 0$ and $x \in \mathrm{dom}\, f^n$ there holds*

$$\mu(f^n(x)) = g^{k(x)+\dots+k(f^{n-1}(x))}(\mu(x))\,.$$

(ii) *If for $x \in \mathrm{dom}\, f$ there exists $n > 0$ such that $x \in \mathrm{dom}\, f^n$ and $f^n(x) = x$, then there holds*

$$g^{k(x)+k(f(x))+\dots+k(f^{n-1}(x))}(\mu(x)) = \mu(x)\,.$$

(iii) *$\mu(ZA) \subseteq ZB$.*
(iv) *If $RA \neq 0$ then $RB \neq 0$.*
(v) *$SA(x) \leqq SB(\mu(x))$ for any $x \in A$.*

Theorem 1. *Let A, B be c-machines. Then there exists a simulation $\mu : A \to B$ if and only if $(A, B) \in sAD$.*

For the proof of lemma and theorem see [7]. □

4. SIMULATIONS OF PAWLAK MACHINES

Suppose that A is a Pawlak machine. We denote by ΘA the partition $A/\varrho A$ and further

$$\Theta_1 A = \{T \in \Theta A;\ DT \neq \emptyset\},$$
$$\Theta_2 A = \{T \in \Theta A;\ DT = \emptyset\}.$$

Definition. Let A be a Pawlak machine. We put

$$\chi_1 A = \begin{cases} \sup\{\chi_0 T;\ T \in \Theta_1 A\} & \text{if } \Theta_1 A \neq 0, \\ 0 & \text{if } \Theta_1 A = \emptyset; \end{cases}$$

$$\chi_2 A = \begin{cases} \sup\{\chi_0 T;\ T \in \Theta_2 A\} & \text{if } \Theta_2 A \neq \emptyset, \\ 0 & \text{if } \Theta_2 A = \emptyset; \end{cases}$$

$$\chi A = (\chi_1 A, \chi_2 A).$$

Definition. Let A, B be Pawlak machines. Then we put

$$\chi A \leqq \chi B \quad \text{iff} \quad \chi_1 A \leqq \chi_1 B, \quad \chi_2 A \leqq \chi_2 B.$$

Lemma 3. *Let A, B be Pawlak machines. Then there exists a simulation $\mu : A \to B$ iff for any $T_1 \in \Theta A$ there exists $T_2 \in \Theta B$ such that $(T_1, T_2) \in sAD$.*

The assertion follows from theorem 1 and lemma 2(*i*). □

Theorem 2. *Let A, B be Pawlak machines. Then there exists a simulation $\mu : A \to B$ if and only if $\chi A \leqq \chi B$.*

Proof. Necessity. Let there exist a simulation $\mu : A \to B$. Then for any $T_1 \in \Theta A$ there exists $T_2 \in \Theta B$ such that $(T_1, T_2) \in sAD$. If $\Theta_1 A = \emptyset$, then $\chi_1 A = 0 \leqq \chi_1 B$. Let $\Theta_1 A \neq \emptyset$. Let $T_1 \in \Theta_1 A$ be arbitrary. Then there exists $T_2 \in \Theta B$ such that $(T_1, T_2) \in sAD$. Thus $DT_2 \neq \emptyset$, $T_2 \in \Theta_1 B$ and $\chi_0 T_1 \leqq \chi_0 T_2$. Then $\chi_1 A \leqq \chi_1 B$ follows from definition.

Analogously we obtain $\chi_2 A \leqq \chi_2 B$. Hence, together $\chi A \leqq \chi B$.

Sufficiency. Let $\chi A \leqq \chi B$. Then $\chi_1 A \leqq \chi_1 B$, $\chi_2 A \leqq \chi_2 B$. Let $T_1 \in \Theta A$ be arbitrary.

If $T_1 \in \Theta_1 A$, then according to the premise there holds $\chi_0 T_1 \leqq \chi_1 A \leqq \chi_1 B$. Hence, by definition of $\chi_1 B$ there exists $T_2 \in \Theta_1 B$ such that $\chi_0 T_1 \leqq \chi_0 T_2$; i.e. $(T_1, T_2) \in sAD$. Analogously if $T_1 \in \Theta_2 A$. Thus, to arbitrary $T_1 \in \Theta A$ there exists $T_2 \in \Theta B$ such that $(T_1, T_2) \in sAD$. Thus there exists a simulation $\mu : A \to B$. □

5. THE CATEGORY OF PAWLAK MACHINES

Lemma 4. *Let $A = (A, f)$, $B = (B, g)$, $C = (C, h)$ be Pawlak machines, $\mu : A \to B$, $\nu : B \to C$ simulations. Then $\nu . \mu : A \to C$ is also a simulation.*

Proof. Let $x \in \operatorname{dom} f$. Then according to the definition of μ there exists $k(x) > 0$ such that $\mu(x) = y \in \operatorname{dom} g^{k(x)}$ and $\mu(f(x)) = g^{k(x)}(y)$ holds. By definition of ν and lemma 2(i) it follows that $\nu(y) \in \operatorname{dom} h^{k(y)+\dots+k(g^{k(x)-1}(y))}$ and $\nu(g^{k(x)}(y)) = h^{k(y)+\dots+k(g^{k(x)-1}(y))}(\nu(y))$. Hence $\nu(\mu(x)) \in \operatorname{dom} h^{k(y)+\dots+k(g^{k(x)-1}(y))}$ and $\nu(\mu(f(x))) = \nu(g^{k(x)}(y)) = h^{k(y)+\dots+k(g^{k(x)-1}(y))}(\nu(y)) = h^{k(y)+\dots+k(g^{k(x)-1}(y))}(\nu(\mu(x)))$. Thus $\nu . \mu$ is a simulation of A into C. □

Further on we want to study the category of all Pawlak machines where morphisms are simulations. This category is denoted by the symbol $\mathcal{S}$. $\mathcal{S}^c$ means the category of all c-machines.

We can describe categories $\mathcal{S}^c$ and $\mathcal{S}$ by means of $\mathcal{S}^c(b)$ and $\mathcal{S}(b)$.

Definition. Let $\mathcal{S}^c$ be the category of c-machines. We put

$$f - \mathcal{S}^c = \{A \in \mathcal{S}^c;\ DA \neq \emptyset\}\ ;$$

$$\bar{f} - \mathcal{S}^c = \{A \in \mathcal{S}^c;\ DA = \emptyset\}\ .$$

We denote $\mathcal{O}_1$ a thin category such that $\operatorname{ob} \mathcal{O}_1 = 0^i \cup \{d_1\}$ and $(\alpha, \beta)_{\mathcal{O}_1} \neq \emptyset$ iff $\alpha \leqq \beta$ for any $\alpha, \beta \in \operatorname{ob} \mathcal{O}_1$.

Further let $\mathcal{O}_2$ be a thin category such that $\operatorname{ob} \mathcal{O}_2 = 0^l \cup \{d_2, d_3\}$ and $(\gamma, \delta)_{\mathcal{O}_2} \neq \emptyset$ iff $\gamma \leqq \delta$ for any $\gamma, \delta \in \operatorname{ob} \mathcal{O}_2$.

Finally we put $\mathcal{O} = \mathcal{O}_1 + \mathcal{O}_2$.

Let $\lambda \in 0^i \cup 0^l \cup \{d_1, d_2, d_3\}$ be arbitrary. Then we put

$$\lambda - \mathcal{S}^c = \{A \in \mathcal{S}^c;\ \chi_0 A = \lambda\}\ .$$

Lemma 5. (i) $f - \mathcal{S}^c(b) = \sum^l_{\alpha \in \mathcal{O}_1} \alpha - \mathcal{S}^c(b)$;

(ii) $\bar{f} - \mathcal{S}^c(b) = \sum^l_{\beta \in \mathcal{O}_2} \beta - \mathcal{S}^c(b)$;

(iii) $\mathcal{S}^c(b) = f - \mathcal{S}^c(b) + \bar{f} - \mathcal{S}^c(b)$;

(iv) $\mathcal{S}^c(b) = \sum^l_{\lambda \in \mathcal{O}} \lambda - \mathcal{S}^c(b)$, *where the subcategory* $\lambda - \mathcal{S}^c(b)$ *is with non-empty homs for any* $\lambda \in \mathcal{O}$.

Proof. For $A, B \in f - \mathcal{S}^c$ there exists a simulation of A into B iff $\chi_0 A \leqq \chi_0 B$ in $\mathcal{O}_1$. Analogously for $A, B \in \bar{f} - \mathcal{S}^c$. Thus, (i) and (ii) hold. If $A \in f - \mathcal{S}^c$, $B \in \bar{f} - \mathcal{S}^c$ there never exists a simulation of A into B since $DA \neq \emptyset$, $DB = \emptyset$. Analogously vice versa. Thus, (iii) holds. The assertion (iv) is the consequence of the more general formula $\sum^l_{G \in \mathcal{G}_1 + \mathcal{G}_2} \mathcal{A}_G = \sum^l_{G \in \mathcal{G}_1} \mathcal{A}_G + \sum^l_{G \in \mathcal{G}_2} \mathcal{A}_G$. The fact that the subcategory $\lambda - \mathcal{S}^c(b)$ is with non-empty homs for any $\lambda \in \mathcal{O}$ is a consequence of theorem 2. □

We denote by $\mathcal{K}$ the thin category such that $\operatorname{ob} \mathcal{K} = \{(\alpha_1, \alpha_2);\ \alpha_1 \in \{0\} \cup 0^i \cup \{d_1\};\ \alpha_2 \in \{0\} \cup 0^l \cup \{d_2, d_3\}\} - \{(0, 0)\}$. For arbitrary $a = (\alpha_1, \alpha_2)$, $b = (\beta_1, \beta_2) \in \operatorname{ob} \mathcal{K}$, we put $(a, b)_{\mathcal{K}} \neq \emptyset$ iff $\alpha_1 \leqq \beta_1$, $\alpha_2 \leqq \beta_2$. Clearly, $\mathcal{K}$ is an ordered class, and $\chi : \mathscr{S} \to \mathscr{K}$ is an object funktion. It can be proved by induction that $\chi\mathscr{S} = \mathscr{K}$.

Let $a \in \mathscr{K}$ be arbitrary. Then we put

$$a - \mathscr{S} = \{A \in \mathscr{S};\ \chi A = a\}.$$

Theorem 3.

$$\mathscr{S}(b) = \sum_{a \in \mathscr{K}}^{l} a - \mathscr{S}(b),$$

where the subcategory $a - \mathscr{S}(b)$ is with non-empty homs for any $a \in \mathscr{K}$.

Proof. 1. Let $a \in \mathscr{K}$ be arbitrary, $A, B \in a - \mathscr{S}$. Then by theorem 2 there holds $(A, B)\mathscr{S} \neq \emptyset$. Thus, $a - \mathscr{S}$ is the category with non-empty homs.

2. Clearly, $\operatorname{ob} \mathscr{S}(b) = \bigcup_{a \in \mathscr{K}} \operatorname{ob}(a - \mathscr{S}(b))$ with disjoint summands.

Let $A, B \in \mathscr{S}(b)$ be arbitrary, $A \in a - \mathscr{S}(b)$, $B \in a' - \mathscr{S}(b)$ Then by definition of the lexicographic sum, $(A, B)_\Sigma \neq \emptyset$ iff $a < a'$, and $a = a'$ implies $A \prod_{a - \mathscr{S}(b)} B$. This holds iff $(A, B)_{\mathscr{S}(b)} \neq \emptyset$ by theorem 2. □

6. THE BREADTH OF THE CATEGORY OF PAWLAK MACHINES

Let us define the concept of the breadth $b(\mathscr{C})$ for arbitrary category $\mathscr{C}$.

If there exists a proper subclass $\mathscr{C}'$ of $\mathscr{C}$ which is an antichain, then we put $b(\mathscr{C}) = \infty^+$.

If, otherwise, any antichain $\mathscr{C}' \subseteq \mathscr{C}$ is a set, we denote $A_\mathscr{C} = \{m$; there is an antichain $\mathscr{C}' \subseteq \mathscr{C}$ such that $\operatorname{card} \mathscr{C}' = m\}$. If $A_\mathscr{C}$ is not bounded, then we put $b(\mathscr{C}) = \infty$. If it is bounded, we denote $B_\mathscr{C} = \{n;\ n \geqq m$ for any $m \in A_\mathscr{C}\}$. Then we put $b(\mathscr{C}) = \min B_\mathscr{C}$. Compare [8].

Theorem 4. *Let $\mathscr{A} = \sum_{I \in \mathscr{G}}^{l} \mathscr{A}_I$ be a category and $\mathscr{A}_I$ a chain for each $I \in \mathscr{G}$. Then there holds $b(\mathscr{A}) = b(\mathscr{G})$.*

Proof. Let $\mathscr{A} = \sum_{I \in \mathscr{G}}^{l} \mathscr{A}_I$. It means that $\operatorname{ob} \mathscr{A} = \bigcup_{I \in \mathscr{G}} \operatorname{ob} \mathscr{A}_I$ where $\mathscr{A}_I$ are mutually disjoint thin categories and for arbitrary $I_1, I_2 \in \mathscr{G}$ and $K \in \mathscr{A}_{I_1}$, $L \in \mathscr{A}_{I_2}$, the condition $(K, L)_\mathscr{A} \neq \emptyset$ holds iff $I_1 <_\mathscr{G} I_2$ and $I_1 = I_2$ implies $K \prod \mathscr{A}_{I_1} L$. Let $(K, L)_\mathscr{A} = \emptyset$, $K \neq L$; this situation will be denoted by writting $K \parallel_\mathscr{A} L$. That means either $I_1 \parallel_\mathscr{G} I_2$ or $I_1 = I_2$, $(K, L)_{\mathscr{A}_{I_1}} = \emptyset$. The latter part of the condition cannot occur because $b(\mathscr{A}_{I_1}) = 1$. Thus, it follows $K \parallel_\mathscr{A} L$ iff $I_1 \parallel_\mathscr{G} I_2$ and we obtain $b(\mathscr{A}) = b(\mathscr{G})$. □

Lemma 6.

$$b(\mathscr{S}^c) = 2.$$

Proof. It follows from lemma 5.(iv) since $\mathscr{O} = \mathscr{O}_1 + \mathscr{O}_2$ where $\mathscr{O}_1 . \mathscr{O}_2$ are chains. □

Definition. Let $\mathscr{A}$ be an ordered class. Then $N \in \operatorname{ob} \mathscr{A}$ is called the *least object of* $\mathscr{A}$ if $N \leqq_{\mathscr{A}} M$ for any $M \in \operatorname{ob} \mathscr{A}$.

An ordered class $\mathscr{A}$ is called a *well ordered class* if any its non-empty subclass has the least object.

Theorem 5. *Let $\mathscr{A}, \mathscr{B}$ be well ordered classes which have at least $\aleph_0$ objects. Then $b(\mathscr{A} . \mathscr{B}) = \aleph_0$.*

Proof. First we prove that in $\mathscr{C} = \mathscr{A} . \mathscr{B}$ all antichains are finite. Let $P = (P_1, P_2)$, $Q = (Q_1, Q_2) \in \mathscr{C}$ and $(P, Q)_{\mathscr{C}} = \emptyset$. Then there holds either $P_1 <_{\mathscr{A}} Q_1$, $P_2 >_{\mathscr{B}} Q_2$ or $P_1 >_{\mathscr{A}} Q_1$, $P_2 <_{\mathscr{B}} Q_2$. Suppose that in $\mathscr{C}$ there exists an infinite antichain $\mathscr{C}_1 = \{(K_1, L_1), (K_2, L_2) \ldots\}$. Then all K_i are mutually different and L_i as well. Since $\mathscr{A}$ is a well ordered class without loss of generality we can assume that $K_1 <_{\mathscr{A}} K_2 <_{\mathscr{A}} <_{\mathscr{A}} \ldots$. Thus, it follows that $L_1 >_{\mathscr{B}} L_2 >_{\mathscr{B}} \ldots$. But this contradicts the condition that $\mathscr{B}$ is well ordered. We have proved that $b(\mathscr{C}) \leqq \aleph_0$. In the rest we prove that for any $n > 0$ there exists an antichain $\mathscr{C}'$ in $\mathscr{C}$ such that $\operatorname{card} \mathscr{C}' = n$. Let n be arbitrary. Since $\mathscr{A}, \mathscr{B}$ have at least $\aleph_0$ objects, we may choose objects $A_1 <_{\mathscr{A}} <_{\mathscr{A}} A_2 <_{\mathscr{A}} \ldots <_{\mathscr{A}} A_n$ in $\mathscr{A}$ and objects $B_1 >_{\mathscr{B}} B_2 >_{\mathscr{B}} \ldots >_{\mathscr{B}} B_n$ in $\mathscr{B}$. We put $\mathscr{C}' = \{(A_i, B_i); 1 \leqq i \leqq n\}$. Clearly, $\mathscr{C}'$ is an antichain and $\operatorname{card} \mathscr{C}' = n$. □

Corollary.

$$b(\mathscr{S}) = \aleph_0 .$$

Proof. The assertion follows from theorems 3, 4 and 5 because $\mathscr{K} = (\{0\} \oplus \mathscr{O}_1) . . (\{0\} \oplus \mathscr{O}_2) - \{(0, 0)\}$ where both factors are clearly well ordered classes. □

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