COELEVATION OF A GRAPH

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I. INTRODUCTION

Some problems gave rise to the notions of elevation and coelevation of a graph, which were introduced in [1]. The elevation of a graph with regard to the crossing numbers of a certain infinite class of graphs was studied in [2]. The purpose of this paper is to determine the coelevation for certain classes of graphs and to investigate some of its basic properties.

Throughout this paper, by a graph we mean a loopless undirected graph without multiple edges. The concepts undefined here are used in the sense of the monograph [3].

II. DEFINITIONS AND SOME KNOWN RESULTS

Let us recall definitions of the elevation and coelevation of a graph.

To every one-to-one labeling $f : V(G) \to \{1, 2, \ldots, n\}$ one can assign the number

$$\varepsilon_f(G) = \sum_{(v_i, v_j) \in G} |f(v_i) - f(v_j)| ,$$

where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the vertex set of a graph G. Then the number

$$\varepsilon(G) = \min_f \varepsilon_f(G)$$

is called the *elevation* of G, and the number

$$\bar{\varepsilon}(G) = \max_f \varepsilon_f(G)$$

is known as the *coelevation* of G.

We list some of the known results (see [1]), which will be used in what follows:

1) *Let K_n be a complete graph with n vertices. Then*

$$\varepsilon(K_n) = \bar{\varepsilon}(K_n) = \binom{n+1}{3} .$$

2) *Let G be an n-vertex graph and $\bar{G}$ its complement. Then*

$$\varepsilon(G) + \bar{\varepsilon}(\bar{G}) = \varepsilon(K_n) .$$

3) *Let G be a graph and $G = G_1 \cup G_2$. Then*

$$\varepsilon(G) = \varepsilon(G_1) + \varepsilon(G_2) .$$

4) *Let G be a graph and G_i, $i = 1, 2, \ldots, k$, its disjoint factors (i.e. subgraphs of G with the same vertex set set as G), the union of which covers G. If there exists a labeling f with the property*

$$\varepsilon_f(G_i) = \varepsilon(G_i) , \quad i = 1, 2, \ldots, k ,$$

then

$$\varepsilon_f(G) = \varepsilon(G) .$$

III. MAIN RESULTS

First, in contrast to 3) of Section II, we present

Theorem 1. *Let G be a graph of the form $G = G_1 \cup G_2$ with at least one edge. Then*

$$\bar{\varepsilon}(G) > \bar{\varepsilon}(G_1) + \bar{\varepsilon}(G_2) .$$

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_m\}$. Suppose that f_1 and f_2 are arbitrary maximal labelings of G_1 and G_2, respectively. It means

$$\varepsilon_{f_1}(G_1) = \bar{\varepsilon}(G_1) \quad \text{and} \quad \varepsilon_{f_2}(G_2) = \bar{\varepsilon}(G_2) .$$

Without loss of generality, we can assume that $\bar{G}_1$ is not a complete graph. Let $v_i \in V(G_1)$ be such a vertex for which $f_1(v_i) = n$. Then $\deg v_i \geqq 1$ (otherwise, f_1 would not be the maximal labeling of G_1). Further, let $u_j \in V(G_2)$ be such a vertex for which $f_2(u_j) = 1$. Take the following labeling f of G:

$$\begin{aligned} f(v_k) &= f_1(v_k) , \quad \text{if} \quad k \neq i , \\ f(u_k) &= f_2(u_k) + n , \quad \text{if} \quad k \neq j , \quad \text{and} \\ f(v_i) &= n + 1 , \quad f(u_j) = n . \end{aligned}$$

Then

$$\varepsilon_f(G) = \varepsilon_{f_1}(G_1) + \deg v_i + \varepsilon_{f_2}(G_2) + \deg u_j > \varepsilon_{f_1}(G_1) + \varepsilon_{f_2}(G_2) .$$

Because $\bar{\varepsilon}(G) \geqq \varepsilon_f(G)$, Theorem 1 is proved.

As an analogue to 4) of Section II, we obtain

Theorem 2. *Let G be a graph, and G_i, $i = 1, 2, \ldots, k$, its disjoint factors, the union of which covers G. If there exists a labeling f with the property*

$$\varepsilon_f(G_i) = \bar{\varepsilon}(G_i) , \quad i = 1, 2, \ldots, k ,$$

then

$$\varepsilon_f(G) = \bar{\varepsilon}(G) .$$

Proof. The proof follows immediately from the inequalities

$$\varepsilon_f(G) = \sum_{i=1}^{k} \varepsilon_f(G_i) = \sum_{i=1}^{k} \bar{\varepsilon}(G_i) \geqq \bar{\varepsilon}(G) ,$$

$$\bar{\varepsilon}(G) \geqq \varepsilon_f(G) .$$

Now we concentrate our attention on the coelevation of certain classes of graphs.

Proposition 1. *Let $R_1(n)$ be a regular graph of degree one with $n \geqq 2$ vertices. Then*

$$\bar{\varepsilon}(R_1(n)) = \frac{n^2}{4} .$$

Proof. Let $V(R_1(n)) = \{v_1, v_2, \ldots, v_n\}$ and $n = 2k$, $k \geqq 1$. For any labeling f of $R_1(n)$ we get

$$\varepsilon_f(R_1(n)) = \sum_{(v_i, v_j) \in R_1(n)} |f(v_i) - f(v_j)| \leqq \sum_{i=k+1}^{2k} i - \sum_{i=1}^{k} i = k^2 = \frac{n^2}{4} ,$$

because in the sum every number of the set $\{1, 2, \ldots, 2k\}$ occurs exactly once.

Suppose that $R_1(n)$ consists of k edges (v_i, v_{i+k}), $i = 1, 2, \ldots, k$. It is sufficient to take the labeling $f : f(v_j) = j$, to complete the proof.

Corollary 1. *Let $R_{n-2}(n)$ be a regular graph of degree $n - 2$ with $n \geqq 2$ vertices. Then*

$$\varepsilon(R_{n-2}(n)) = \frac{(n-2)\, n(2n+1)}{12} .$$

Proof. If we put $G = R_{n-2}(n)$, then $\bar{G} = R_1(n)$. Clearly, 1), 2) and Proposition 1 imply the statement.

Proposition 2. *Let $R_2(n)$ be a regular graph of degree two with $n \geqq 3$ vertices. Then*

$$\bar{\varepsilon}(R_2(n)) \leqq \frac{n^2}{2} \text{ if } n \text{ is even} ,$$

$$\bar{\varepsilon}(R_2(n)) \leqq \frac{n^2 - 1}{2} \text{ if } n \text{ is odd} .$$

Proof. Take an arbitrary labeling $f : f(v_i) = x_i$ of the vertices $v_1, v_2, \ldots, v_n$ of

$R_2(n)$. Compute:

$$\varepsilon_f(R_2(n)) = \sum_{(v_a,v_b)\in R_2(n)} |f(v_a) - f(v_b)| = \sum_{\alpha=1}^{n} x_{j_\alpha} - \sum_{\beta=1}^{n} x_{j_\beta} \,.$$

Among all numbers $x_{j_\alpha}, x_{j_\beta}$, every number of the set $\{1, 2, \ldots, n\}$ occurs exactly twice. Hence, for any $k \geqq 2$ we have

$$\varepsilon_f(R_2(n)) \leqq 2 \sum_{i=k+1}^{2k} i - 2\sum_{i=1}^{k} i = 2k^2 = \frac{n^2}{2} \quad \text{if} \quad n = 2k\,,$$

$$\varepsilon_f(R_2(n)) \leqq 2 \sum_{i=k+1}^{2k-1} i - 2\sum_{i=1}^{k-1} i = 2k^2 - 2k = \frac{n^2-1}{2} \quad \text{if} \quad n = 2k-1\,.$$

Since f was an arbitrary labeling, it follows that

$$\bar{\varepsilon}(R_2(n)) \leqq \frac{n^2}{2} \quad \text{if } n \text{ is even}\,,$$

$$\bar{\varepsilon}(R_2(n)) \leqq \frac{n^2-1}{2} \quad \text{if } n \text{ is odd}\,.$$

The upper bounds are attained for a cycle C_n with $n \geqq 3$ vertices. As a maximal labeling f of C_n for both $n = 2k$ and $n = 2k - 1$ we can take the following one:

$$f(v_{2i-1}) = i\,, \quad f(v_{2i}) = n - (i-1)\,, \quad i = 1, 2, \ldots, k\,,$$

where the vertices $v_1, v_2, \ldots, v_n$ are denoted in cylic order. This completes the proof.

Proposition 2 yields several consequences.

Corollary 2. *For the cycle C_n we have*

$$\bar{\varepsilon}(C_n) = \frac{n^2}{2} \quad \text{if } n \text{ is even}\,,$$

$$\bar{\varepsilon}(C_n) = \frac{n^2-1}{2} \quad \text{if } n \text{ is odd}\,.$$

Proof. If we take the labeling of C_n used in the proof of Proposition 2, the proof of Corollary 2 follows immediately.

Corollary 3. *Let P_n be a path with n vertices. Then*

$$\bar{\varepsilon}(P_n) = \frac{n^2-2}{2} \quad \text{if } n \text{ is even}\,,$$

$$\bar{\varepsilon}(P_n) = \frac{n^2-3}{2} \quad \text{if } n \text{ is odd}\,.$$

Proof. To get a maximal labeling of P_n, it is sufficient to take the labeling of C_n used in the proof of Proposition 2, and to remove the edge which represents the difference of value one. With regard to Corollary 2, the statement is proved.

Corollary 4. *Let $R_{n-3}(n)$ be a regular graph of degree $n - 3$ with $n \geqq 3$ vertices. Then*

$$\varepsilon(R_{n-3}(n)) \geqq \frac{n^3 - 3n^2 - n}{6} \quad \text{if } n \text{ is even },$$

$$\varepsilon(R_{n-3}(n)) \geqq \frac{(n^2 - 1) \cdot (n - 3)}{6} \quad \text{if } n \text{ is odd }.$$

The lower bounds are attained if $\overline{R_{n-3}(n)} = C_n$.

Proof. Let us put $G = R_{n-3}(n)$. Then $\bar{G} = R_2(n)$, and 1) and 2) with respect to Proposition 2 and Corollary 2 prove the above statement.

Proposition 3. *Let $K_n \cup K_n$ be a graph consisting of two disjoint copies of K_n, $n \geqq 2$. Then*

$$\bar{\varepsilon}(K_n \cup K_n) = 4\binom{n+1}{3}.$$

Proof. Let us have an arbitrary labeling f of $K_n \cup K_n$. Denote the values of the vertices in the form of increasing sequences

$$x_1 < x_2 < \ldots < x_n ,$$

$$y_1 < y_2 < \ldots < y_n ,$$

for each graph K_n separately. Suppose that n is an even number. We compute

$$\varepsilon_f(K_n \cup K_n) = \sum_{\substack{p>r \\ r=1,p=2}}^{n} (x_p - x_r) + \sum_{\substack{p>r \\ r=1,p=2}}^{n} (y_p - y_r) =$$

$$= \sum_{i=1}^{n/2} (n - 2i + 1) \cdot (x_{n-i+1} - x_i) + \sum_{i=1}^{n/2} (n - 2i + 1) \cdot (y_{n-i+1} - y_i) =$$

$$= \sum_{i=1}^{n/2} (n - 2i + 1) \cdot [(x_{n-i+1} + y_{n-i+1}) - (x_i + y_i)] .$$

If we use the obvious inequalities

$$4i - 1 \leqq x_i + y_i \leqq 2i + n \quad \text{if} \quad i \leqq \frac{n}{2},$$

$$2i + n \leqq x_i + y_i \leqq 4i - 1 \quad \text{if} \quad i \geqq \frac{n}{2} + 1 ,$$

we obtain

$$\varepsilon_f(X_n \cup K_n) \leqq \sum_{i=1}^{n/2} (n - 2i + 1) . [(4n - 4i + 3) - (4i - 1)] =$$

$$= 4 \sum_{i=1}^{n/2} (n - 2i + 1)^2 = 4 \binom{n+1}{3} .$$

Because f was an arbitrary labeling of $K_n \cup K_n$ we have

$$\bar{\varepsilon}(K_n \cup K_n) \leqq 4 \binom{n+1}{3} .$$

For n odd the proof is analogous, therefore it is omitted. As a maximal labeling f of $K_n \cup K_n$ we can take

$$f : x_i = 2i - 1 , \quad y_i = 2i , \quad i = 1, 2, \ldots, n .$$

Hence, the proposition is proved.

Corollary 5. *Let $K_{n,n}$ be a complete bipartite graph. Then*

$$\varepsilon(K_{n,n}) = \frac{n . (2n^2 + 1)}{3} .$$

Proof. Let us put $G = K_{n,n}$. Then $\bar{G} = K_n \cup K_n$, and 1), 2) and Proposition 3 imply the Corollary.

In the end, we state one property of an arbitrary maximal labeling of a graph.

Theorem 3. *Let G be a graph with n vertices and let f be one of its maximal labelings. Then for a vertex v such that $f(v) = n$ we have*

$$\sum_{i=1}^{k} f(x_i) \leqq \frac{n}{2} . \deg v ,$$

where the vertices $x_1, x_2, \ldots, x_k$ are incident with the vertex v.

Proof. We do it indirectly. Assume that

$$\sum_{i=1}^{k} f(x_i) > \frac{n}{2} . \deg v .$$

Compute the sum

$$\sum_{i=1}^{k} [f(v) - f(x_i)] = n . \deg v - \sum_{i=1}^{k} f(x_i) < \sum_{i=1}^{k} f(x_i) .$$

If we use the following labeling g:

$$g(u) = f(u) + 1 \quad \text{if} \quad u \neq v ,$$
$$g(v) = 1 ,$$

then

$$\varepsilon_g(G) = \sum_{\substack{(a,b)\in G\\ a\neq v, b\neq v}} |g(a) - g(b)| + \sum_{i=1}^{k} [g(x_i) - g(v)] =$$

$$= \sum_{\substack{(a,b)\in G\\ a\neq v, b\neq v}} |f(a) - f(b)| + \sum_{i=1}^{k} f(x_i) >$$

$$> \sum_{\substack{(a,b)\in G\\ a\neq v, b\neq v}} |f(a) - f(b)| + \sum_{i=1}^{k} [f(v) - f(x_i)] = \varepsilon_f(G)$$

yields a contradiction.

References

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