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Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 1, 1–6

Persistent URL: <http://dml.cz/dmlcz/101849>

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ON A CERTAIN NUMBERING OF THE VERTICES OF A HYPERGRAPH

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(Received May 5, 1979)

0. By a *hypergraph* we shall mean an ordered pair $\mathcal{H} = (V, \mathcal{E})$, where V is a finite nonempty set, and \mathcal{E} is a set of nonempty subsets of V (note that our concept of a hypergraph is not identical with the concept of a hypergraph in the sense of [1]). The elements of V are called *vertices* of \mathcal{H} and the elements of \mathcal{E} are called *edges* of \mathcal{H} .

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Denote $n = |V|$. Consider a sequence (v_1, \dots, v_n) such that $\{v_1, \dots, v_n\} = V$. If for each $E \in \mathcal{E}$ there exist integers i and k , $1 \leq i \leq k \leq n$, with the property that

$$E = \{v_j; i \leq j \leq k\},$$

then we shall say that the sequence (v_1, \dots, v_n) is a *projectoidic arrangement* of \mathcal{H} . Obviously, if (v_1, \dots, v_n) is a projectoidic arrangement of \mathcal{H} , then the sequence (v_n, \dots, v_1) is also a projectoidic one. We shall say that \mathcal{H} is a *projectoid* if there exists a projectoidic arrangement of \mathcal{H} . This means that \mathcal{H} is a projectoid if and only if its vertices can be numbered by the integers $1, \dots, n$ in such a way that for each $E \in \mathcal{E}$, if i, j , and k are integers, $1 \leq i \leq j \leq k \leq n$, such that both i and k are the numbers assigned to some vertices of E , then j is also the number assigned to a vertex of E .

Objects equivalent to projectoids were studied by means of the matrix theory in [3] and [7], and by means of the theory of bipartite graphs in [7]. As families of sets projectoids were studied in [2] and [6] (an applications of projectoids in the area of information retrieval was shown in [2]). In [2], [3], [6], and [7] various characterizations for projectoids (or objects equivalent to them) can be found. For the full list of "subhypergraphs" (in a certain sense) which are forbidden for projectoids the reader is referred to [6]. (Note that the terms "projectoidic" or "projectoid" have not appeared in the papers mentioned above).

It is obvious that a hypergraph with at most two edges is a projectoid. In the present paper for every hypergraph \mathcal{H} we shall construct a certain set of hypergraphs with exactly three edges and show that \mathcal{H} is a projectoid if and only if each hypergraph

in the constructed set is. The proof of this is based on the concept of a strict separating set (see below). In the last section of the paper this result will be applied to a problem concerning directed graphs.

1.1. Let \mathcal{A} be a finite nonempty set of finite nonempty sets. Then we denote by $\langle \mathcal{A} \rangle$ the hypergraph (V', \mathcal{A}) , where

$$V' = \bigcup_{A \in \mathcal{A}} A .$$

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. If $\mathcal{A} \subseteq \mathcal{E}$, then instead of $(V, \mathcal{E} - \mathcal{A})$ we shall write $\mathcal{H} - \mathcal{A}$. If Z is a nonempty subset of V , then we denote by $\langle Z \rangle_{\mathcal{H}}$ the hypergraph (Z, \mathcal{E}') , where

$$\mathcal{E}' = \{E \cap Z; E \in \mathcal{E} \text{ and } E \cap Z \neq \emptyset\} .$$

We denote by $\Omega(\mathcal{H})$ the set defined as follows:

- (1) if $v \in V$, then $\{v\} \in \Omega(\mathcal{H})$;
- (2) if $E \in \mathcal{E}$, then $E \in \Omega(\mathcal{H})$;
- (3) if $S', S'' \in \Omega(\mathcal{H})$ and $S' \cap S'' \neq \emptyset$, then $S' \cup S'' \in \Omega(\mathcal{H})$;
- (4) no other element belongs to $\Omega(\mathcal{H})$.

It follows from (1) that the hypergraphs $\langle \Omega(\mathcal{H}) \rangle$ and $(V, \Omega(\mathcal{H}))$ are identical. It is obvious that there exists exactly one partition \mathcal{P} of V with the properties that (a) if $U \in \mathcal{P}$, then $U \in \Omega(\mathcal{H})$; and (b) if $E \in \mathcal{E}$, then there exists $W \in \mathcal{P}$ such that $E \subseteq W$. If $V' \in \mathcal{P}$, then we shall say that $\langle V' \rangle_{\mathcal{H}}$ is a component of \mathcal{H} . We say that \mathcal{H} is connected if it has exactly one component. Clearly, \mathcal{H} is connected if and only if $V \in \Omega(\mathcal{H})$. Let $\mathcal{A} \subseteq \mathcal{E}$; we say that \mathcal{A} is a separating set of \mathcal{H} if $\mathcal{H} - \mathcal{A}$ is not connected. We say that a separating set \mathcal{A} of \mathcal{H} is strict if no proper subset of \mathcal{A} is a separating set of \mathcal{H} .

Proofs of the following four propositions will be left to the reader:

Proposition 1. *Let $\mathcal{H} = (V, \mathcal{E})$ be a projectoid, and let $V' \subseteq V$ and $\mathcal{E}' \subseteq \mathcal{E}$, where $V' \neq \emptyset \neq \mathcal{E}'$. Then both $\langle V' \rangle_{\mathcal{H}}$ and $\langle \mathcal{E}' \rangle$ are projectoids.*

Proposition 2. *Let \mathcal{H} be a hypergraph. Then every projectoidic arrangement of \mathcal{H} is a projectoidic arrangement of $\langle \Omega(\mathcal{H}) \rangle$.*

Proposition 3. *A hypergraph \mathcal{H} is a projectoid if and only if $\langle \Omega(\mathcal{H}) \rangle$ is.*

Proposition 4. *Let S_1, S_2 , and S_3 be three finite nonempty sets. Then $\langle \{S_1, S_2, S_3\} \rangle$ is a projectoid if and only if the following conditions hold:*

- (1) *if there exists a permutation p on $\{1, 2, 3\}$ such that $S_{p(1)} \cap (S_{p(2)} - S_{p(3)}) \neq \emptyset \neq S_{p(1)} \cap (S_{p(3)} - S_{p(2)})$, then $S_{p(2)} \cap S_{p(3)} \subseteq S_{p(1)}$;*
- (2) *if the sets $S_1 \cap S_2$, $S_2 \cap S_3$, and $S_3 \cap S_1$ are nonempty, then there exists a permutation q on $\{1, 2, 3\}$ such that $S_{q(1)} \subseteq S_{q(2)} \cup S_{q(3)}$.*

We now state the main result of this paper:

Theorem 1. *Let \mathcal{H} be a hypergraph. Then it is a projectoid if and only if for any three elements $S_1, S_2,$ and S_3 of $\Omega(\mathcal{H}), \langle\{S_1, S_2, S_3\}\rangle$ is a projectoid.*

1.2. Proof of Theorem 1. Denote $\mathcal{H} = (V, \mathcal{E})$ and $|V| = n$.

(A) Assume that \mathcal{H} is a projectoid. According to Proposition 3, $\langle\Omega(\mathcal{H})\rangle$ is a projectoid. It follows from Proposition 1 that for any three $S_1, S_2, S_3 \in \Omega(\mathcal{H}), \langle\{S_1, S_2, S_3\}\rangle$ is a projectoid.

(B) Assume that for any three $S_1, S_2, S_3 \in \Omega(\mathcal{H}), \langle\{S_1, S_2, S_3\}\rangle$ is a projectoid. We shall prove that \mathcal{H} is a projectoid.

It follows from assumption (B) that

(*) for any nonempty proper subset V' of V and for any three $S'_1, S'_2, S'_3 \in \Omega(\langle V' \rangle_{\mathcal{H}}), \langle\{S'_1, S'_2, S'_3\}\rangle$ is a projectoid.

If $n \leq 2$, then \mathcal{H} is a projectoid. Let $n \geq 3$. Assume that for every hypergraph $\mathcal{H}' = (V', \mathcal{E}')$ with $|V'| < n$ and with the property that

for every three $S'_1, S'_2, S'_3 \in \Omega(\mathcal{H}'), \langle\{S'_1, S'_2, S'_3\}\rangle$ is a projectoid,

it has been proved that \mathcal{H}' is a projectoid. It follows from (*) and from the induction assumption that

for every nonempty proper subset V' of $V, \langle V' \rangle_{\mathcal{H}}$ is a projectoid.

If $V \in \mathcal{E}$, then \mathcal{H} is a projectoid if and only if $(V, \mathcal{E} - \{V\})$ is a projectoid. Therefore, without loss of generality we shall assume that $V \notin \mathcal{E}$. We distinguish the following cases:

(1) Assume that \mathcal{H} is not connected. Then every component of \mathcal{H} is a projectoid. Hence, \mathcal{H} is also a projectoid.

(2) Assume that \mathcal{H} is connected.

(2.1) Assume that for every strict separating set \mathcal{A} of \mathcal{H} , there exists a vertex of \mathcal{H} , say a vertex $r(\mathcal{A})$, such that $\langle V - \{r(\mathcal{A})\} \rangle_{\mathcal{H}}$ is a component of $\mathcal{H} - \mathcal{A}$. Since $n \geq 3$, we have that $r(\mathcal{A})$ is determined uniquely.

Let \mathcal{B} be an arbitrary strict separating set of \mathcal{H} . If $B_1, B_2 \in \mathcal{B}$, then from the fact that $\langle\{B_1, B_2, V - \{r(\mathcal{B})\}\}\rangle$ is a projectoid it follows according to Proposition 4 that either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence, \mathcal{B} is linearly ordered by the inclusion. We denote by \mathcal{B}^* the minimum edge of \mathcal{B} . We have that \mathcal{B} is the set of edges $E \in \mathcal{E}$ with the properties that $r(\mathcal{B}) \in E$ and $|E| \geq 2$. This implies that if \mathcal{B}' is a strict separating set of \mathcal{H} , then $\mathcal{B} = \mathcal{B}'$ if and only if $r(\mathcal{B}) = r(\mathcal{B}')$.

Consider a strict separating set \mathcal{U} of \mathcal{H} . Since $V \notin \mathcal{E}$, there exists a strict separating set \mathcal{W} of \mathcal{H} such that $\mathcal{U}^* \notin \mathcal{W}$. For every strict separating set \mathcal{A} of \mathcal{H} , either $\mathcal{A} = \mathcal{U}$ or $\mathcal{A} = \mathcal{W}$ (otherwise, $\langle\{V - \{r(\mathcal{A})\}, V - \{r(\mathcal{U})\}, V - \{r(\mathcal{W})\}\}\rangle$ is not a projectoid, which is a contradiction). This implies that $\mathcal{U}^* \cup \mathcal{W}^* = V$ and $\mathcal{U}^* \cap \mathcal{W}^* \neq \emptyset$. Assume that there exists $X \in \mathcal{U} \cap \mathcal{W}$. Then $\mathcal{U}^* \subseteq X$ and $\mathcal{W}^* \subseteq X$. Hence, $X = V$. Thus $V \in \mathcal{E}$, which is a contradiction. This means that $\mathcal{U} \cap \mathcal{W} = \emptyset$.

Without loss of generality we shall assume that $|\mathcal{U}^*| \geq |\mathcal{W}^*|$. It is obvious that $\langle V - \{r(\mathcal{U})\} \rangle_{\mathcal{H}}$ is a projectoid. We denote by (v_1, \dots, v_{n-1}) a projectoidic arrange-

ment of $\langle V - \{r(\mathcal{U})\} \rangle_{\mathcal{H}}$. Since $|\mathcal{U}^* - \{r(\mathcal{U})\}| \leq n - 2$ and $\mathcal{H}^* \in \mathcal{E}$, we have that either $v_1 \notin \mathcal{U}^*$ or $v_{n-1} \notin \mathcal{U}^*$. Without loss of generality we assume that $v_{n-1} \notin \mathcal{U}^*$. Hence, $v_{n-1} \in \mathcal{H}^*$. If $|\mathcal{H}^*| = n - 1$, then $|\mathcal{U}^*| = n - 1$, and therefore, $v_1 \in \mathcal{U}^*$. Let $|\mathcal{H}^*| \leq n - 2$; since $v_{n-1} \in \mathcal{H}^*$ and $\mathcal{H}^* \in \mathcal{E}$, we have that $v_1 \notin \mathcal{H}^*$; hence, $v_1 \in \mathcal{U}^*$. This means that (u, v_1, \dots, v_{n-1}) is a projectoidic arrangement of \mathcal{H} . Therefore, \mathcal{H} is a projectoid.

(2.2) Assume that there exists a strict separating set \mathcal{A} of \mathcal{H} such that the hypergraph $\mathcal{H} - \mathcal{A}$ contains no component with $n - 1$ vertices.

(2.2.1) Assume that $\mathcal{H} - \mathcal{A}$ has at least three components. Let $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$, $\mathcal{H}_2 = (V_2, \mathcal{E}_2), \dots, \mathcal{H}_k = (V_k, \mathcal{E}_k)$ be the components of $\mathcal{H} - \mathcal{A}$. Hence, $k \geq 3$. Since \mathcal{A} is a strict separating set of \mathcal{H} , we have that for every i , $1 \leq i \leq k$, and every $A \in \mathcal{A}$, the inequality $A \cap V_i \neq \emptyset$ holds.

Assume that for every j , $1 \leq j \leq k$, there exists $A_j \in \mathcal{A}$ such that $V_j - A_j \neq \emptyset$. Denote

$$B_1 = A_1 \cup V_2 \cup V_3, \quad B_2 = A_2 \cup V_3 \cup V_1, \quad B_3 = A_3 \cup V_1 \cup V_2.$$

Clearly, $B_1, B_2, B_3 \in \Omega(\mathcal{H})$. We can see that

$$V_3 - A_3 \subseteq B_1 \cap (B_2 - B_3), \quad V_2 - A_2 \subseteq B_1 \cap (B_3 - B_2),$$

and

$$V_1 - A_1 \subseteq (B_2 \cap B_3) - B_1.$$

Since $V_j - A_j \neq \emptyset$, for $1 \leq j \leq 3$, it follows from Proposition 4 that $\langle \{B_1, B_2, B_3\} \rangle$ is not a projectoid, which is a contradiction. This means that there exists f , $1 \leq f \leq k$, such that for every $A \in \mathcal{A}$, $V_f \subseteq A$.

Let $(u_1, \dots, u_{n-|V_f|})$ be a projectoidic arrangement of $\langle V - V_f \rangle_{\mathcal{H}}$. There exists g , $1 \leq g \leq k$ and $g \neq f$, such that $u_1 \in V_g$. Clearly, $u_1, \dots, u_{|V_g|} \in V_g$. Let $(w_1, \dots, w_{|V_f|})$ be a projectoidic arrangement of $\langle V_f \rangle_{\mathcal{H}}$. Then

$$(u_1, \dots, u_{|V_g|}, w_1, \dots, w_{|V_f|}, u_{|V_g|+1}, \dots, u_{n-|V_f|})$$

is a projectoidic arrangement of \mathcal{H} . Hence, \mathcal{H} is a projectoid.

(2.2.2) Assume that $\mathcal{H} - \mathcal{A}$ has exactly two components, say the components $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$. Obviously, $\min(|V_1|, |V_2|) \geq 2$. Since \mathcal{A} is a strict separating set of \mathcal{H} , we have for every $A \in \mathcal{A}$ the inequalities $A \cap V_1 \neq \emptyset \neq A \cap V_2$. Consider arbitrary $A', A'' \in \mathcal{A}$. Since both $\langle \{V_1, A' \cup V_2, A'' \cup V_2\} \rangle$ and $\langle \{V_2, A' \cup V_1, A'' \cup V_1\} \rangle$ are projectoids, we have that (a) either $A' \cap V_1 \subseteq A'' \cap V_1$ or $A'' \cap V_1 \subseteq A' \cap V_1$, and (b) $A' \cap V_2 \subseteq A'' \cap V_2$ or $A'' \cap V_2 \subseteq A' \cap V_2$. This implies that there exists $v_1 \in V_1$ and $v_2 \in V_2$ such that for every $A \in \mathcal{A}$, we have $v_1, v_2 \in A$.

Consider a projectoidic arrangement $(u_0, \dots, u_{|V_1|})$ of $\langle V_1 \cup \{v_2\} \rangle_{\mathcal{H}}$, and a projectoidic arrangement $(w_0, \dots, w_{|V_2|})$ of $\langle V_2 \cup \{v_1\} \rangle_{\mathcal{H}}$. It is clear that without loss of generality we may assume that $u_{|V_1|} = v_2$ and $w_0 = v_1$. It is not difficult to see that

$(u_0, \dots, u_{|V|-1}, w_1, \dots, w_{|V|-1})$ is a projectoidic arrangement of \mathcal{H} . Hence, \mathcal{H} is a projectoid, which completes the proof of Theorem 1.

2. Let $D = (V, A)$ be a digraph in the sense of [4]. For every $v \in V$, we denote by $R(v, D)$ the set of vertices which are reachable from v (in D). Obviously, $w \in R(w, D)$, for each $w \in V$. Denote

$$\mathcal{R}(D) = \{R(v, D); v \in V\}.$$

We denote by $[D]$ the graph obtained from D in such a way that each arc (u, v) is replaced by the edge $\{u, v\}$. If $u, v, w \in V$, then we shall say that v is (u, w) -reachable (in D) if for every path P (in the sense of [3]) which connects u with w in $[D]$, there exists a vertex t_P belonging to P and such that $v \in R(t_P, D)$.

Let $D = (V, A)$ be a digraph. Denote $|V| = n$. Consider a sequence (v_1, \dots, v_n) such that $\{v_1, \dots, v_n\} = V$. We shall say that the sequence (v_1, \dots, v_n) is a *projective arrangement* of D if it is a projectoidic arrangement of the hypergraph $(V, \mathcal{R}(D))$. The term “projective” in the sense of the present paper has its origin in mathematical linguistics, namely in studying sentence structures. For some further details the reader is referred to [5].

We shall say that a digraph D is a *project* if there exists a projective arrangement of D . It is obvious that a digraph $D = (V, A)$ is a project if and only if $(V, \mathcal{R}(D))$ is a projectoid. For example, every out-tree is a project. There exists exactly one digraph with less than five vertices which is not a project; it is the in-tree T with the property that $[T]$ is the star $K_{1,3}$.

The proof of the following proposition is easy (cf. the proof of Theorem 3.2 in [5]).

Proposition 5. *Let (v_1, \dots, v_n) be a projective arrangement of a project D . Then for any three integer i, j , and k , $1 \leq i \leq j \leq k \leq n$, v_j is (v_i, v_k) -reachable.*

The following theorem is a solution of the problem which was stated by the present author at Czechoslovak Graph Theory Conference held in Brno, May 1975:

Theorem 2. *Let $D = (V, A)$ be a digraph. Then it is a project if and only if for any $v_1, v_2, v_3 \in V$, there exists a permutation p on $\{1, 2, 3\}$ such that $v_{p(2)}$ is $(v_{p(1)}, v_{p(3)})$ -reachable.*

Proof. One of the implications in the statement of Theorem 2 follows immediately from Proposition 5. We shall prove the other one.

Let D not be a project. Then $(V, \mathcal{R}(D))$ is not a projectoid. According to Theorem 1, there exist distinct $S_1, S_2, S_3 \in \Omega((V, \mathcal{R}(D)))$ such that $\langle \{S_1, S_2, S_3\} \rangle$ is not a projectoid. We distinguish two cases:

(1) Assume that the set $S_1 - (S_2 \cup S_3)$, $S_2 - (S_3 \cup S_1)$, and $S_3 - (S_1 \cup S_2)$ are nonempty. Consider $v_1 \in S_1 - (S_2 \cup S_3)$, $v_2 \in S_2 - (S_3 \cup S_1)$, and $v_3 \in S_3 - (S_1 \cup S_2)$. Since $S_1, S_2, S_3 \in \Omega((V, \mathcal{R}(D)))$, we have that v_i , where $i = 1, 2, 3$, is not reachable from any vertex in S_j , where $j = 1, 2, 3$ and $j \neq i$. Since $\langle \{S_1, S_2, S_3\} \rangle$ is not a projectoid, it follows from Proposition 4 that $S_1 \cap S_2$, $S_2 \cap S_3$, and $S_3 \cap S_1$

are nonempty. This means that in $[D]$ there exist paths P_{12} , P_{23} , and P_{31} which connect v_1 with v_2 , v_2 with v_3 , and v_3 with v_1 , respectively, such that each vertex of P_{12} , P_{23} , and P_{31} , belongs to $S_1 \cup S_2$, $S_2 \cup S_3$, and $S_3 \cup S_1$, respectively. Hence, for any permutation p on $\{1, 2, 3\}$, $v_{p(2)}$ is not $(v_{p(1)}, v_{p(3)})$ -reachable.

(2) Assume that at least one of the sets $S_1 - (S_2 \cup S_3)$, $S_2 - (S_3 \cup S_1)$, and $S_3 - (S_1 \cup S_2)$ is empty. Without loss of generality we assume that $S_1 \subseteq S_2 \cup S_3$. If $S_1 \cap (S_2 - S_3) = \emptyset$ or $S_1 \cap (S_3 - S_2) = \emptyset$, then $S_1 \subseteq S_3$ or $S_1 \subseteq S_2$, respectively, and therefore, $\langle \{S_1, S_2, S_3\} \rangle$ is a projectoid, which is a contradiction. This means that $S_1 \cap (S_2 - S_3) \neq \emptyset \neq S_1 \cap (S_3 - S_2)$. It follows from Proposition 4 that $(S_2 \cap S_3) - S_1 \neq \emptyset$. Consider $v_{12} \in S_1 \cap (S_2 - S_3)$, $v_{13} \in S_1 \cap (S_3 - S_2)$, and $v_{23} \in (S_2 \cap S_3) - S_1$. It is clear that v_{12} , v_{13} and v_{23} are reachable from no vertex in S_3 , S_2 and S_1 , respectively. There exist paths P_1 , P_2 , and P_3 which connect v_{12} with v_{13} , v_{12} with v_{23} , and v_{13} with v_{23} , respectively, such that each vertex of P_1 , P_2 , and P_3 , belongs to S_1 , S_2 , and S_3 , respectively. Hence, for any permutation p on $\{1, 2, 3\}$, $v_{p(2)}$ is not $(v_{p(1)}, v_{p(3)})$ -reachable.

Thus the proof of Theorem 2 is complete.

I wish to express my sincere thanks to Dr. Z. Starý for his comments on the text of this paper.

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