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ALMOST PRODUCT RIEMANNIAN MANIFOLDS

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**Introduction and notation.** An almost product manifold is a smooth  $m$ -manifold  $M$  with a smooth tensor field  $F$  of type  $(1,1)$  such that  $F^2 = I_m$ . If  $M$  has a positive definite Riemannian metric  $g$  such that  $g(\bar{X}, \bar{Y}) = g(X, Y)$ , (where  $X, Y$  are arbitrary vector fields and  $\bar{X} = FX$  throughout the paper) then  $M$  is called an *almost product Riemannian manifold* or *almost para Hermite manifold*, [1], [2], [5]. If  $(D_X F) Y = 0$  for all  $X, Y$  then  $M$  is called *para Kahler*.  $D$  is the natural metric connection. The manifold  $M$  will be called *para Hermite* if  $(D_X F) Y \neq 0$  and there exist real constants  $K_1, K_2, \dots, K_7$ , not all zeros such that:

$$(*) \quad (D_X F) Y + K_1(D_X F) Y + K_2(D_X F) \bar{Y} + K_3(D_X F) \bar{Y} + K_4(D_Y F) X + K_5(D_Y F) \bar{X} + K_6(D_Y F) X + K_7(D_Y F) \bar{X} = 0.$$

(\*) will be called a *para Hermite condition*.

In this paper we will determine all classes of para Hermite manifolds and discuss inclusions among them. In the last section we discuss the relation between integrability and parallelism of distributions on  $M$  as related to para Hermite structures. I would like to thank professor R. S. Mishra for the useful discussions.

**Proposition 1.** *Let  $(M, g, F)$  be an almost product Riemannian manifold. If  $F(X, Y) = g(\bar{X}, Y)$  then:*

- i)  $F'(X, Y) = F'(Y, X) = F'(\bar{X}, \bar{Y})$ ,
- ii)  $(D_X F')(Y, Z) = g((D_X F) Y, Z) = g(Y, (D_X F) Z)$ ,
- iii)  $(D_X F')(Y, Z) = -(D_X F')(\bar{Y}, \bar{Z})$ ,
- iv)  $(D_X F')(\bar{Y}, Z) = -(D_X F')(Y, \bar{Z})$ ,
- v)  $(D_X F')(Y, Z) = (D_X F')(Z, Y)$

where  $X, Y, Z$  are arbitrary vector fields.

The proof of this proposition is easy and will be omitted.

**Theorem 1.** *The following is a para Hermite condition:*

$$(P_1) \quad (D_X F) Y - (D_Y F) X = 0.$$

A para Hermite manifold satisfying  $(P_1)$  will be called  $P_1$ -Hermite.

Proof. Define  $I = (D_X F) Y$ ,  $\sigma = (D_Y F) X$ , then  $\{I, \sigma\}$  admits the following table of multiplication:

Table 1

	$I$	$\sigma$
$I$	$I$	$\sigma$
$\sigma$	$\sigma$	$I$

Thus the set  $G_1$  of all linear combinations of  $I, \sigma$  is an infinite commutative ring under addition and multiplication as defined in table 1. Now in  $G_1$  we have:

$$I = \sigma^2, \quad (I - \sigma)(I + \sigma) = 0.$$

Thus we have one of the following possibilities:

i)  $I - \sigma = 0$ ,  $I + \sigma \neq 0$ , then  $(D_X F) Y - (D_Y F) X = 0$  and the manifold is  $P_1$ -Hermite.

ii)  $I - \sigma \neq 0$ ,  $I + \sigma = 0$ , then  $(D_X F) Y + (D_Y F) X = 0$  or  $(D_X F) Y = -(D_Y F) X$ .

Using Proposition 1 we get:

$$\begin{aligned} (D_X F')(Y, Z) &= -(D_Y F')(X, Z) = -(D_Y F')(Z, X) = \\ &= (D_Z F')(X, Y) = -(D_X F')(Y, Z). \end{aligned}$$

Thus  $(D_X F')(Y, Z) = 0$ . Since  $Z$  is arbitrary then using Proposition 1 again we get  $(D_X F) Y = 0$  and  $I + \sigma = 0$  is equivalent to the para Kähler condition, i.e. it is not a para Hermite condition.

iii)  $I - \sigma = 0$ ,  $I + \sigma = 0$ , is not a para Hermite condition as can be seen from (ii) above.

iv)  $I - \sigma \neq 0$ ,  $I + \sigma \neq 0$ , the manifold is not para Hermite.

Since  $I - \sigma = 0$  is a para Hermite condition then  $(I - \sigma)^n = 0$ ,  $n \in N$  will give para Hermite conditions. But  $(I - \sigma)^2 = 0 \Leftrightarrow I - 2\sigma + I = 0 \Leftrightarrow 2(I - \sigma) = 0$  which is the  $P_1$ -Hermite condition. So is  $(I - \sigma)^n = 0$ ,  $n \in N$ . Thus we conclude that in  $G_1$ , the only para Hermite condition is the one obtained.

**Theorem 2.** *The following are para Hermite conditions:*

- |         |  |                  |
|---------|--|------------------|
| $(P_2)$ | $(D_X F) Y - (D_X F) \bar{Y} = 0,$                   | $P_2$ -Hermite , |
| $(P_3)$ | $(D_X F) Y + (D_X F) \bar{Y} = 0,$                   | $P_3$ -Hermite , |
| $(P_4)$ | $(D_X F) Y - (D_X F) \bar{Y} = 0,$                   | $P_4$ -Hermite , |
| $(P_5)$ | $(D_X F) Y + (D_X F) \bar{Y} = 0,$                   | $P_5$ -Hermite , |
| $(P_6)$ | $(D_X F) Y - (D_X F) \bar{Y} = 0,$                   | $P_6$ -Hermite , |
| $(P_7)$ | $(D_X F) Y + (D_X F) \bar{Y} = 0,$                   | $P_7$ -Hermite , |
| $(P_8)$ | $(D_X F) Y - (D_X F) \bar{Y} + (D_X F) \bar{Y} = 0,$ | $P_8$ -Hermite , |

- ( $P_9$ )  $(D_X F) Y - (D_{\bar{X}} F) Y + (D_X F) \bar{Y} - (D_{\bar{X}} F) \bar{Y} = 0$ ,  $P_9$ -Hermite ,  
 ( $P_{10}$ )  $(D_X F) Y + (D_{\bar{X}} F) Y - (D_X F) \bar{Y} - (D_{\bar{X}} F) \bar{Y} = 0$ ,  $P_{10}$ -Hermite ,  
 ( $P_{11}$ )  $(D_X F) Y + (D_{\bar{X}} F) Y + (D_X F) \bar{Y} + (D_{\bar{X}} F) \bar{Y} = 0$ ,  $P_{11}$ -Hermite .

Proof. Let  $I = (D_X F) Y$ ,  $\alpha = (D_{\bar{X}} F) Y$ ,  $\beta = (D_X F) \bar{Y}$ ,  $\gamma = (D_{\bar{X}} F) \bar{Y}$ . Then  $I, \alpha, \beta, \gamma$  admit the following table of multiplication:

Table 2

	$I$	$\alpha$	$\beta$	$\gamma$
$I$	$I$	$\alpha$	$\beta$	$\gamma$
$\alpha$	$\alpha$	$I$	$\gamma$	$\beta$
$\beta$	$\beta$	$\gamma$	$I$	$\alpha$
$\gamma$	$\gamma$	$\beta$	$\alpha$	$I$

The set  $G_2$  of all linear combinations of  $I, \alpha, \beta, \gamma$  will be an infinite commutative ring under addition and multiplication as in table 2. Now in  $G_2$  we have:

$$\begin{aligned}
 I &= \alpha^2, & (I - \alpha)(I + \alpha) &= 0, \\
 I &= \beta^2, & (I - \beta)(I + \beta) &= 0, \\
 I &= \gamma^2, & (I - \gamma)(I + \gamma) &= 0,
 \end{aligned}$$

Arguments similar to those in theorem 1 will give  $P_i$ -Hermite,  $i = 2, \dots, 7$ . Since products of para Hermite conditions give para Hermite conditions, we take products of conditions (2) to (7) on the following table:

Table 3

	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$	$P_7$	$P_8$	$P_9$	$P_{10}$	$P_{11}$
$P_2$	$P_2$	0	$P_8$	$P_9$	$P_9$	$P_8$	$P_8$	$P_9$	0	0
$P_3$		$P_3$	$P_{10}$	$P_{11}$	$P_{10}$	$P_{11}$	0	0	$P_{10}$	$P_{11}$
$P_4$			$P_4$	0	$P_{10}$	$P_8$	$P_8$	0	$P_{10}$	0
$P_5$				$P_5$	$P_9$	$P_{11}$	0	$P_9$	0	$P_{11}$
$P_6$					$P_6$	0	0	$P_9$	$P_{10}$	0
$P_7$						$P_7$	$P_8$	0	0	$P_{11}$
$P_8$							$P_8$	0	0	0
$P_9$								$P_9$	0	0
$P_{10}$									$P_{10}$	0
$P_{11}$										$P_{11}$

Thus from table 3 we see that multiplications of  $P_2$  to  $P_7$  yield

- ( $P_8$ )  $(I - \alpha)(I - \beta) = 0$ ,  $P_8$ -Hermite ,  
 ( $P_9$ )  $(I - \alpha)(I + \beta) = 0$ ,  $P_9$ -Hermite ,

$$(P_{10}) \quad (I + \alpha)(I - \beta) = 0, \quad P_{10}\text{-Hermite},$$

$$(P_{11}) \quad (I + \alpha)(I + \beta) = 0, \quad P_{11}\text{-Hermite}$$

and that  $P_2$  to  $P_{11}$  are the only structures that can be obtained in  $G_2$ .

Now, in order to get more structures, we should be able to multiply elements of  $G_1$  by those of  $G_2$ . This can be done if  $G_1, G_2$  are subrings of a larger ring  $G$ .

$G$  can be constructed in four different ways according to how we interpret the operations  $\alpha, \beta, \gamma, \sigma$ .

**Case 1.** Let  $\alpha, \beta, \gamma$  bar  $X, Y, X$  and  $Y$  respectively and  $\sigma$  switches slots. In this case the set  $G_3$  of all linear combinations of  $I, \alpha, \beta, \gamma, \sigma, \sigma\alpha, \sigma\beta, \sigma\gamma$  is a commutative ring with multiplication defined by tables 1, 2 and the following:

$$(A) \quad \sigma\alpha = \alpha\sigma = (D_Y F)\bar{X}, \quad \sigma\beta = \beta\sigma = (D_Y F)X, \quad \sigma\gamma = \gamma\sigma = (D_Y F)\bar{X}.$$

**Theorem 3.** *The following are para Hermite conditions:*

$$(P_{12}) \quad (D_X F)Y - (D_Y F)X - (D_X F)Y + (D_Y F)\bar{X} = 0, \quad P_{12}\text{-Hermite},$$

$$(P_{13}) \quad (D_X F)Y - (D_Y F)X + (D_X F)Y - (D_Y F)\bar{X} = 0, \quad P_{13}\text{-Hermite},$$

$$(P_{14}) \quad (D_X F)Y - (D_Y F)X - (D_X F)\bar{Y} + (D_Y F)X = 0, \quad P_{14}\text{-Hermite},$$

$$(P_{15}) \quad (D_X F)Y - (D_Y F)X + (D_X F)\bar{Y} - (D_Y F)X = 0, \quad P_{15}\text{-Hermite},$$

$$(P_{16}) \quad (D_X F)Y - (D_Y F)X - (D_X F)\bar{Y} + (D_Y F)\bar{X} = 0, \quad P_{16}\text{-Hermite},$$

$$(P_{17}) \quad (D_X F)Y - (D_Y F)X + (D_X F)\bar{Y} - (D_Y F)\bar{X} = 0, \quad P_{17}\text{-Hermite},$$

$$(P_{18}) \quad (D_X F)Y - (D_X F)Y - (D_X F)\bar{Y} + (D_X F)\bar{Y} - (D_Y F)X + \\ + (D_Y F)\bar{X} + (D_Y F)X - (D_Y F)\bar{X} = 0, \quad P_{18}\text{-Hermite},$$

$$(P_{19}) \quad (D_X F)Y - (D_X F)Y + (D_X F)\bar{Y} - (D_X F)\bar{Y} - (D_Y F)X + \\ + (D_Y F)\bar{X} - (D_Y F)X + (D_Y F)\bar{X} = 0, \quad P_{19}\text{-Hermite},$$

$$(P_{20}) \quad (D_X F)Y + (D_X F)Y - (D_X F)\bar{Y} - (D_X F)\bar{Y} - (D_Y F)X - \\ - (D_Y F)\bar{X} + (D_Y F)X + (D_Y F)\bar{X} = 0, \quad P_{20}\text{-Hermite},$$

$$(P_{21}) \quad (D_X F)Y + (D_X F)Y + (D_X F)\bar{Y} + (D_X F)\bar{Y} - (D_Y F)X - \\ - (D_Y F)\bar{X} - (D_Y F)X - (D_Y F)\bar{X} = 0, \quad P_{21}\text{-Hermite}.$$

*Proof.* Conditions 12 to 21 are given respectively by:

$$(I - \sigma)(I - \alpha), (I - \sigma)(I + \alpha), (I - \sigma)(I - \beta), (I - \sigma)(I + \beta),$$

$$(I - \sigma)(I - \gamma), (I - \sigma)(I + \gamma), (I - \sigma)(I - \alpha - \beta + \gamma),$$

$$(I - \sigma)(I - \alpha + \beta - \gamma), (I - \sigma)(I + \alpha - \beta - \gamma), (I - \sigma)(I + \alpha + \beta + \gamma).$$

From the above construction and theorems 1 and 2 we see that these are the only structures in  $G_3$ .

**Case 2.** In this case we let  $\alpha, \beta, \gamma$  bar first, second, first and second slots respectively and  $\sigma$  switches vectors. The set  $G_4$  of all linear combinations of  $I, \alpha, \beta, \gamma, \sigma, \sigma\alpha, \sigma\beta, \sigma\gamma$  is an infinite commutative ring with  $G_1, G_2$  as subrings. In  $G_4$  product is given by Tables 1, 2 and the following:

$$(B) \quad \sigma\alpha = \alpha\sigma = (D_Y F) X, \quad \sigma\beta = \beta\sigma = (D_Y F) \bar{X}, \quad \sigma\gamma = \gamma\sigma = (D_Y F) \bar{X}.$$

**Theorem 4.** *The following are para Hermite conditions*

$$(P_{22}) \quad (D_X F) Y - (D_Y F) X - (D_{\bar{X}} F) Y + (D_Y F) X = 0, \quad P_{22}\text{-Hermite},$$

$$(P_{23}) \quad (D_X F) Y - (D_Y F) X + (D_{\bar{X}} F) Y - (D_Y F) X = 0, \quad P_{23}\text{-Hermite},$$

$$(P_{24}) \quad (D_X F) Y - (D_Y F) X - (D_X F) \bar{Y} + (D_Y F) \bar{X} = 0, \quad P_{24}\text{-Hermite},$$

$$(P_{25}) \quad (D_X F) Y - (D_Y F) X + (D_X F) \bar{Y} - (D_Y F) \bar{X} = 0, \quad P_{25}\text{-Hermite}.$$

Proof. These are respectively given by  $(I - \sigma)(I - \alpha), (I - \sigma)(I + \alpha), (I - \sigma) \cdot (I - \beta), (I - \sigma)(I + \beta)$ .

The rest of the products give structures already given in theorem 3.

**Case 3.** Let  $\alpha, \beta, \gamma$  bar vectors and  $\sigma$  switches vectors. Here we have:

$$(C) \quad \sigma\alpha = \beta\sigma = (D_Y F) X, \quad \alpha\sigma = \sigma\beta = (D_Y F) \bar{X}, \quad \sigma\gamma = \gamma\sigma = (D_Y F) \bar{X}.$$

If we take  $G_5$  to be the set of all linear combinations of  $I, \alpha, \beta, \gamma, \sigma, \sigma\alpha, \alpha\sigma, \sigma\gamma$ , then  $G_5$  will be an infinite non commutative ring with  $G_1, G_2$  as subrings.

To get new structures in  $G_5$  we have to multiply on both sides. It can be seen easily from A, B and C that if we multiply in one direction, we obtain the structures of theorem 3, and if we multiply in the other direction we get those of theorem 4. Thus in this case no new structures are obtained.

**Case 4.** Here  $\alpha, \beta, \gamma$  bar slots and  $\sigma$  switches slots and we get:

$$\alpha\sigma = \sigma\beta = (D_Y F) X, \quad \sigma\alpha = \beta\sigma = (D_Y F) \bar{X}, \quad \sigma\gamma = \gamma\sigma = (D_Y F) \bar{X}$$

and as in case 3 nothing new is obtained.

**Inclusions.** Now, let us study inclusions among these classes of para Hermite structures. The class of  $P_i$ -Hermite manifolds is contained in the class of  $P_j$ -Hermite manifolds ( $P_i \subset P_j$ ) if

$$(P_i) \Rightarrow (P_j)$$

where  $(P_i), (P_j)$  represent the para Hermite conditions.

**Theorem 5.** *In the class of para Hermite manifolds the following inclusion relations hold:*

$$P_1 \subset P_6, \quad P_2, P_4, P_7 \subset P_8, \quad P_2, P_5, P_6 \subset P_9,$$

$$P_3, P_4, P_6 \subset P_{10}, \quad P_3, P_5, P_7 \subset P_{11},$$

$$\begin{aligned}
P_1, P_2 &\subset P_{12}, & P_1, P_2 &\subset P_{22}, \\
P_1, P_3 &\subset P_{13}, & P_1, P_3 &\subset P_{23}, \\
P_1, P_4 &\subset P_{14}, & P_1, P_4 &\subset P_{24}, \\
P_1, P_5 &\subset P_{15}, & P_1, P_5 &\subset P_{25}, \\
P_1, P_6 &\subset P_{16}, & P_1, P_7 &\subset P_{17}, & P_1, P_8 &\subset P_{18}, \\
P_1, P_9 &\subset P_{19}, & P_1, P_{10} &\subset P_{20}, & P_1, P_{11} &\subset P_{21}.
\end{aligned}$$

Proof. We will only prove that  $P_1 \subset P_6$ . The rest follow directly from the definitions of the structures.

Since  $(D_X F^2) = (D_X F) F + F(D_X F)$ , and  $F^2 = I_m$  we get:

$$(a) \quad (D_X F) \bar{Y} = -\overline{(D_X F) Y},$$

for all vector fields  $X, Y$  and hence

$$(b) \quad (D_X F) \bar{Y} = -\overline{(D_X F) Y},$$

if  $(P_1)$  holds, i.e.  $(D_X F) Y - (D_Y F) X = 0$ , then from (b) we get:

$$(D_X F) \bar{Y} = -\overline{(D_Y F) X} = \overline{(D_Y F) X} = (D_Y F) X = (D_X F) Y$$

and  $(D_X F) Y - (D_X F) \bar{Y} = 0$  which is  $(P_6)$ .

**Distributions.** It is well known that the existence of an almost product structure  $F$  on a Riemannian manifold  $(M, g)$  is equivalent to the existence of two complementary distributions  $P, Q$  on  $M$ . That is, starting with  $F$ , then  $P$  and  $Q$  can be respectively defined by the following projections:

$$L_1 = \frac{1}{2}[I_m + F], \quad L_2 = \frac{1}{2}[I_m - F].$$

Conversely, starting with two complementary distributions  $P, Q$  on  $M$ , then  $F$  can be defined in a natural way as follows:  $F(X + Y) = X - Y$  where  $X + Y$  is any vector field. In fact, we can start with one distribution  $P$  and take  $Q$  its orthogonal complement.

In the following theorems, which give a geometric explanation of some of the para Hermite structures found,  $F$  is always constructed as above. The proofs of these theorems are direct applications of the results of Walker [3] and Willmore [4] and will be omitted.

**Theorem 6.** *Let  $P$  be a distribution on a Riemannian manifold  $(M, g)$ . If  $(M, g, F)$  is  $P_{11}$ -Hermite then  $P$  is a semi-parallel.*

**Theorem 7.** *Let  $P, Q$  be two complementary distributions on a Riemannian manifold  $(M, g)$ . If  $(M, g, F)$  is  $P_9$ -Hermite then one of the distributions is parallel along the other.*

**Theorem 8.** Let  $P, Q$  be two complementary distributions on a Riemannian manifold  $(M, g)$ . If  $(M, g, F)$  is  $P_7$ -Hermite then both distributions are semi parallel.

**Theorem 9.** Let  $P$  be a distribution on a Riemannian manifold  $(M, g)$ . If  $(M, g, F)$  is  $P_5$ -Hermite then  $P$  is parallel and hence integrable.

**Theorem 10.** Let  $P, Q$  be two complementary distributions on a Riemannian manifold  $(M, g)$ . If  $(M, g, F)$  is para Kähler then both distributions are parallel and hence integrable.

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