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ON K-RADICAL CLASSES OF LATTICE ORDERED GROUPS

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The notion of radical class of lattice ordered groups was introduced in the author’s paper [4]; cf. also [5]. A radical class $X$ which can be defined by means of the properties of the lattice of closed convex $l$-subgroups of $l$-groups belonging to $X$ is said to be a $K$-radical class (P. Conrad [1]).

Examples of $K$-radical classes are: the class $X_A$ of all archimedean $l$-groups; the class $X_B$ of all $l$-groups with a basis; the class $X_C$ of all completely distributive $l$-groups; the class $S_l$ of all $l$-groups $G$ such that the lattice of all closed convex $l$-subgroups of $G$ is a Stone lattice. These and some other examples of $K$-radical classes were thoroughly studied by Conrad [1].

Let $\mathcal{R}$ and $\mathcal{R}_K$ be the collections of all radical classes and $K$-radical classes, respectively. Both $\mathcal{R}$ and $\mathcal{R}_K$ are partially ordered by inclusion. Then $\mathcal{R}$ and $\mathcal{R}_K$ are complete lattices. The lattice $\mathcal{R}$ was investigated in [4]. In the present paper the lattice $\mathcal{R}_K$ will be dealt with.

For $X \in \mathcal{R}_K$ we denote by $a(X)$ the class of all elements of $\mathcal{R}_K$ covering $X$; further, let $a_1(X)$ be the class of all $Y \in \mathcal{R}_K$ such that $X < Y$ and no element of the interval $[X, Y]$ covers $X$. Let $R_0$ and $\mathcal{G}$ be the least element and the greatest element of $\mathcal{R}_K$, respectively. If $X \in \mathcal{R}_K$ is generated by a one-element class, then $X$ is said to be principal.

Sample results: It will be shown that $\mathcal{R}_K$ fails to be a closed sublattice of $\mathcal{R}$. It will be proved that $\mathcal{R}_K$ is a Brouwer lattice and for each subclass $X$ of $\mathcal{G}$ the $K$-radical class generated by $X$ is equal to Join Lat Sub $X$ (for denotations, cf. Sec. 1 below). There exist $K$-radical classes $X_1, X_2$ distinct from $\mathcal{R}_0$ such that $a(X_1) = 0$ and $a_1(X_2) = 0$. If $X \neq R_0$ is a principal $K$-radical class, then both $a(X)$ and $a_1(X)$ are nonempty (in fact, they are proper classes). For each $X \in \{X_A, X_B, X_C, S_l\}$ the class $a(X)$ is nonempty. $K$-radical classes generated by linearly ordered groups will be examined.

1. PRELIMINARIES

Let $\mathcal{G}$ be the class of all lattice ordered groups. When considering a subclass $Y$ of $\mathcal{G}$ we always assume that $Y$ is closed with respect to isomorphisms and that the zero group $\{0\}$ belongs to $Y.$
A subclass $X$ of $\mathcal{G}$ is said to be a **radical class** if it is closed with respect to

a) convex $l$-subgroups, and

b) joins of convex $l$-subgroups.

Let $G \in \mathcal{G}$. We denote by $c(G)$ the system of all convex $l$-subgroups of $G$; the set $c(G)$ is partially ordered by inclusion. Then $c(G)$ is a complete lattice. The operation $\wedge$ in $c(G)$ coincides with the set-theoretical intersection. The join in $c(G)$ will be denoted by $\vee_{i \in I} G_i$.

For $G \in \mathcal{G}$ we denote by $K(G)$ the system of all closed convex $l$-subgroups of $G$. The system $K(G)$ is partially ordered by inclusion; then $K(G)$ is a complete lattice. The lattice operations in $K(G)$ will be denoted by $\wedge$ and $\vee$. For $M \subseteq G$ we denote by $M^-$ the closed convex $l$-subgroup of $G$ that is generated by $M$. The meet in $K(G)$ coincides with the set-theoretical intersection and for $\{G_i\}_{i \in I} \subseteq K(G)$ we have

$$\bigvee_{i \in I} G_i = (\bigvee_{i \in I} G_i)^-.$$

A subclass $X$ of $\mathcal{G}$ is said to be a **$K$-class** if there exists a class $T$ of lattices such that the equivalence

$$G \in X \iff K(G) \subseteq T$$

is valid for each lattice ordered group $G$.

A $K$-class $X$ which is at the same time a radical class is called a **$K$-radical class** (cf. [1]).

Let $\mathcal{R}$ and $\mathcal{R}_K$ be the collections of all radical classes and all $K$-radical classes, respectively. Both $\mathcal{R}$ and $\mathcal{R}_K$ are partially ordered by inclusion. The one-element class $R_0 = \{\{0\}\}$ is the least element in both $\mathcal{R}$ and $\mathcal{R}_K$, and $\mathcal{G}$ is the largest element in both $\mathcal{R}$ and $\mathcal{R}_K$.

For $X \subseteq \mathcal{R}$ we denote by

- **Sub** $X$ — the class of all convex $l$-subgroups of $l$-groups belonging to $X$;
- **Join** $X$ — the class of all $l$-groups $G$ having a system $\{G_i\}_{i \in I}$ of closed convex $l$-subgroups with $G_i \in X$ for each $i \in I$ such that $\bigvee_{i \in I} G_i = G$;
- **Join$_e$** $X$ — the class of all $l$-groups $G$ having a system $\{G_i\}_{i \in I}$ of convex $l$-subgroups with $G_i \in X$ for each $i \in I$ such that $\bigvee_{i \in I} G_i = G$;
- **Lat** $X$ — the class of all lattice ordered groups $G$ such that $K(G)$ is isomorphic to $K(G_1)$ for some $G_1 \in X$;
- $(X)^-$ — the class of all lattice ordered groups $G$ having the property that there exists $G_1 \in c(G) \cap X$ such that $G = G_1^-$.

$\mathcal{R}$ is a complete lattice in which the meet coincides with the intersection of classes. (In fact, $\mathcal{R}$ is a proper class.) The join in $\mathcal{R}$ will be denoted by $\vee^c$. For $X \subseteq \mathcal{G}$ we denote by $T(X)$ the intersection of all $Y \in \mathcal{R}$ with $X \subseteq Y$. Then $T(X)$ belongs to $\mathcal{R}$ and is said to be the **radical class generated by $X$**. The following two propositions were proved in [4].

**1.1. Proposition.** Let $X \subseteq \mathcal{G}$. Then $T(X) = \text{Join}_e \text{Sub } X$. 

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1.2. Proposition. Let I be a nonempty class and for each \( i \in I \) let \( X_i \in \mathcal{R} \). Then \( \bigvee_{i \in I} X_i = \text{Join}(\bigcup_{i \in I} X_i) \).

2. THE LATTICE \( \mathcal{R}_K \)

2.1. Proposition. \( \mathcal{R}_K \) is a complete lattice. The operation of meet in \( \mathcal{R}_K \) coincides with the intersection of classes.

Proof. Let I be a nonempty class and for each \( i \in I \) let \( X_i \) be a \( K \)-radical class. Hence for each \( i \in I \) there exists a class \( T_i \) of lattices such that

\[ G \in X_i \iff K(G) \in T_i \]

is valid for each \( G \in \mathcal{C} \).

Put \( X = \bigcap_{i \in I} X_i \), \( T = \bigcap_{i \in I} T_i \). Then \( X \) is a radical class and for each \( H \in \mathcal{C} \) we have \( H \in X \iff K(H) \in T \); hence \( X \in \mathcal{R}_K \). Thus \( X \) is the meet of the class \( \{X_i\}_{i \in I} \) in \( \mathcal{R}_K \). Because \( \mathcal{R}_K \) is bounded, it is a complete lattice.

2.2. Lemma. Let \( G \in \mathcal{C} \), \( H \in \mathcal{C}(G) \). For each \( H_1 \in K(H) \) and each \( H'_1 \in K(H^-) \) put \( \varphi(H_1) = H_1^-, \psi(H'_1) = H \cap H'_1 \). Then \( \varphi \) is an isomorphism of the lattice \( K(H) \) onto \( K(H^-) \) and \( \psi = \varphi^{-1} \).

Proof. Let \( H_1, H_2 \in K(H) \) and assume that \( \varphi(H_1) = \varphi(H_2) \). Let \( 0 < g \in H_2 \). Thus \( g \in H_2^- = H_1^- \) and hence there are elements \( \{x_i\}_{i \in I} \) in \( H_1 \) such that \( 0 < x_i \) and sup \( \{x_i\} = g \) holds in \( G \). This implies that sup \( \{x_i\} = g \) holds in \( H \) as well and thus, because of \( H_1 \in K(H) \), we have \( g \in H_1 \). Therefore the mapping \( \varphi \) is a monomorphism.

Let \( H^* \in K(H^-) \). Then clearly \( H^* \cap H \in K(H) \) and \( \varphi(H^* \cap H) \subseteq H \). Let \( 0 < g \in H \). There is a subset \( \{x_i\}_{i \in I} \subseteq H \) with \( 0 < x_i \) such that \( \bigvee x_i = g \) holds in \( G \). Then \( \{x_i\}_{i \in I} \subseteq H^* \cap H \) and hence \( g \in \varphi(H^* \cap H) \). Therefore \( \varphi(H^* \cap H) = H^* \). Hence \( \varphi \) is onto \( K(H^-) \). At the same time we have verified that \( \psi = \varphi^{-1} \).

If \( H_1, H_2 \in K(H) \) and \( H_1^*, H_2^* \in K(H^-) \), then

\[ H_1 \subseteq H_2 \Rightarrow \varphi(H_1) \subseteq \varphi(H_2) \]

and

\[ H_1^* \subseteq H_2^* \Rightarrow \psi(H_1^*) \subseteq \psi(H_2^*) \]

hence \( \varphi \) is an isomorphism.

2.3. Lemma. Let \( X \subseteq \mathcal{C} \), \( Y = \text{Lat Sub} \ X \). Then \( \text{Sub} \ Y = Y \).

Proof. Let \( G \in Y \). There exists \( G' \in \text{Sub} \ X \) such that the lattice \( K(G) \) is isomorphic to \( K(G') \); let \( \varphi \) be the corresponding isomorphism. Let \( H \in \mathcal{C}(G) \). Put \( H^* = \varphi(H^-) \).

The lattice \( K(H^*) \) coincides with the interval \([0, H^*]\) of the lattice \( K(G') \); similarly,
the lattice $K(H^-)$ coincides with the interval $[\{0\}, H^-]$ of the lattice $K(G)$. Hence $K(H^-)$ is isomorphic to $K(H^*)$. Because of $H^* \in \text{Sub } X$ we infer that $H^- \in \text{Lat Sub } X$ and in view of 2.2, $H \in Y$. Therefore $\text{Sub } Y = Y$.

2.4. Lemma. Let $X \subseteq \mathcal{G}$, $Y = \text{Join Sub } X$. Then $\text{Sub } Y = Y$.

Proof. Let $H \in \text{Sub } Y$. There is $G \in Y$ with $H \in c(G)$. Further, there are $G_i \in \text{Sub } X \cap K(G)$ with $G = \bigvee_{i \in I} G_i$. Hence $G \cap H \subseteq \text{Sub } X \cap K(G)$. It suffices to verify that $H = \bigvee_{i \in I} (G_i \cap H)$ is valid.

Let $0 < h \in H$. Since $h \in G = (\bigvee_{i \in I} G_i)^\rightarrow$ there are elements $h_j (j \in J)$ such that $h = \bigvee_{j \in J} h_j$ and $0 < h_j \in \bigvee_{i \in I} G_i$ for each $j \in J$.

Let $j \in J$ be fixed. There exist elements $g_{j_1}, \ldots, g_{n(j)} \in \bigcup_{i \in I} G_i$ such that $0 < g_{j_k}$ is valid for $k = 1, 2, \ldots, n(j)$ and

$$h_j = g_{j_1} + \cdots + g_{j_n(j)}.$$ Then $g_{j_k} \leq h_j$ holds for $k = 1, 2, \ldots, n(j)$ and hence $g_{j_k} \in \bigcup_{i \in I} (G_i \cap H)$. Therefore all $h_j$ belong to $\bigvee_{i \in I} (G_i \cap H)$ and so $h \in (\bigvee_{i \in I} (G_i \cap H))^\rightarrow = \bigvee_{i \in I} (G_i \cap H)$. Thus $H = \bigvee_{i \in I} (G_i \cap H)$.

2.5. Lemma. Let $X \subseteq \mathcal{G}$, $Z = \text{Join Lat Sub } X$. Then $\text{Sub } Z = Z$.

Proof. In view of 2.3 we have

$$Z = \text{Join Sub Lat Sub } X;$$

hence according to 2.4 the relation $\text{Sub } Z = Z$ is valid.

2.6. Lemma. Let $G$ and $G'$ be the lattice ordered groups and let $\varphi$ be an isomorphism of $K(G)$ onto $K(G')$. Let $H \in K(G)$. Then $K(H)$ is isomorphic to $K(\varphi(H))$.

Proof. The image of the interval $[\{0\}, H]$ of $K(G)$ under the isomorphism $\varphi$ is the interval $[\{0\}, \varphi(H)]$ of the lattice $K(G')$. Since $K(H) = [\{0\}, H]$ and $K(\varphi(H)) = [\{0\}, \varphi(H)]$, the lattice $K(H)$ is isomorphic to $K(\varphi(H))$.

2.7. Lemma. Let $X \subseteq \mathcal{G}$, $Y = \text{Join Lat } X$. Then $\text{Lat } Y = Y$.

Proof. Let $G \in \text{Lat } Y$. There is $G' \in Y$ such that there exists an isomorphism $\varphi$ of $K(G)$ onto $K(G')$. Further, there are $G'_i (i \in I)$ in $K(G') \cap \text{Lat } X$ such that $G' = \bigvee_{i \in I} G'_i$. Put $G_i = \varphi^{-1}(G'_i)$. Then in view of 2.6 we have $G_i \in K(G) \cap \text{Lat } X$; clearly $G = \bigvee_{i \in I} G_i$, hence $G \in Y$.

2.8. Lemma. Let $X \subseteq \mathcal{G}$, $Z = \text{Join Lat Sub } X$. Then $Z$ is closed with respect to joins of convex l-subgroups.

Proof. Let $G \in \mathcal{G}$ and $G_i \in c(G) \cap Z (i \in I)$. Put $H = \bigvee_{i \in I} G_i$. Then we have $H^- = \bigvee_{i \in I} G_i^-$. According to 2.2 and 2.7, all $G_i^-$ belong to $Z$ and hence $H^- \in \text{Join } Z = Z$. Since $H \in \text{Sub } \{H^-\}$, from 2.5 we infer that $H \in Z$ is valid.
2.9. Theorem. Let \( X \subseteq \mathcal{G} \). Then
(i) Join Lat Sub \( X \in \mathcal{R}_K \);
(ii) for each \( Z \in \mathcal{R}_K \) with \( X \subseteq Z \) we have Join Lat Sub \( X \subseteq Z \).

Proof. Put Join Lat Sub \( X = Z \). In view of 2.5, \( Z \) is closed with respect to convex \( l \)-subgroups; according to 2.8, \( Z \) is also closed with respect to joins of convex \( l \)-subgroups. Therefore \( Z \) is a radical class. Since \( Z \) is a \( K \)-class (cf. 2.7), we have \( Z \in \mathcal{R}_K \). Hence (i) is valid.

Let \( Z' \in \mathcal{R}_K \) be \( X \subseteq Z' \). Since \( Z' \) is a radical class, we have Sub \( Z' = Z' \); because \( Z \) is a \( K \)-class, Lat \( Z' = Z' \) is valid. Hence we have to verify that Join \( Z' = Z' \) holds.

Let \( G \in \mathcal{G} \), \( \{G_i\}_{i \in I} \subseteq X \cap K(G) \), \( \bigvee_{i \in I} G_i = G \). Put \( \bigvee_{i \in I} G_i = H \). In view of 2.8, \( H \in Z' \). Since \( H = G \), from 2.2 we infer that \( G \in Z' \). Thus (ii) holds true.

For each \( X \subseteq \mathcal{G} \) we denote \( T_\mathcal{K}(X) = \text{Join Lat Sub} \ X \). In view of 2.9, \( T_\mathcal{K}(X) \) is the \( K \)-radical class generated by \( X \). If \( G \in \mathcal{G} \), then the \( K \)-radical class generated by \( \{G\} \) will be denoted by \( T_\mathcal{K}(G) \).

The join in the lattice \( \mathcal{R}_K \) will be denoted by \( \vee \).

2.10. Theorem. Let \( I \) be a nonempty class and for each \( i \in I \) let \( X_i \) be a \( K \)-radical class. Then \( \bigvee_{i \in I} X_i = \text{Join} \bigcup_{i \in I} X_i \).

Proof. From 2.9 we infer that
\[
\bigvee_{i \in I} X_i = \text{Join Lat Sub} \bigcup_{i \in I} X_i
\]
is valid. Further, we have
\[
\text{Lat Sub} \bigcup_{i \in I} X_i = \bigcup_{i \in I} \text{Lat Sub} X_i = \bigcup_{i \in I} X_i,
\]
completing the proof.

Let us remark that for \( \{X_i\}_{i \in I} \subseteq \mathcal{R}_K \) the relation
\[
\bigvee_{i \in I} X_i = \bigvee_{i \in I} X_i
\]
need not be valid. This will be shown by Example 3.8 below.

2.11. Theorem. The lattice \( \mathcal{R}_K \) fulfils the infinite distributive law
\[
X \land (\bigvee_{i \in I} Y_i) = \bigvee_{i \in I} (X \land Y_i).
\]

Proof. We have \( \bigvee_{i \in I} (X \land Y_i) \subseteq X \land (\bigvee_{i \in I} Y_i) \). Let \( G \in X \land (\bigvee_{i \in I} Y_i) \). Hence \( G \in X \) and \( G \in \bigvee_{i \in I} Y_i \). In view of 2.10 there are \( G_j (j \in J) \) belonging to \( K(G) \cap \bigcap (\bigcup_{i \in I} Y_i) \) such that
\[
G = \bigvee_{j \in J} G_j.
\]

Then we have \( G_j \in \bigcup_{i \in I} (X \land Y_i) \) for each \( j \in J \), hence \( G \in \bigvee_{i \in I} (X \land Y_i) \).
3. K-RADICAL CLASSES GENERATED BY LINEARLY ORDERED GROUPS

3.1. Proposition. Let $X \subseteq G$. Assume that $\text{Sub } X = X$ and $\text{Lat } X = X$. Then $T_k(X) = (T(X))^\sim$.

Proof. a) Let $G \in (T(X))^\sim$. Hence there is $G_1 \in \mathcal{C}(G) \cap T(X)$ such that $G = G_1^\sim$. In view of 1.1 there are $G_i \in X \cap \mathcal{C}(G_i)$ such that $G_1 = \bigvee_{i \in I} G_i$. Hence $G = G_i^\sim = \bigvee G_i^\sim$ and according to 2.2, $G_i^\sim \in X$. Therefore $G \in T_k(X)$ and thus $(T(X))^\sim \subseteq T_k(X)$.

b) Assume that $G \in T_k(X)$. Hence $G \in \text{Join Lat } X = \text{Join } X$. Thus there are $G_i \in X \cap \mathcal{K}(G)$ with $G = \bigvee_{i \in I} G_i$. Put $H = \bigvee_{i \in I} G_i$; we have $H \in T(X)$ and $H^\sim \in (T(X))^\sim$. From $H^\sim = \bigvee_{i \in I} G_i^\sim = \bigvee_{i \in I} G_i = G$ we infer that $G \in (T(X))^\sim$ and therefore $T_k(X) \subseteq (T(X))^\sim$.

3.2. Lemma. Let $G \in G$. Then the following conditions are equivalent:

(i) $G$ is linearly ordered.

(ii) $\mathcal{C}(G)$ is a chain.

(iii) $\mathcal{K}(G)$ is a chain.

The proof is simple, it will be omitted.

Let us recall the notion of the completely subdirect product of lattice ordered groups which was introduced by F. Sik [7].

Let $G \in G$ and let $S = \{G_i\}_{i \in I}$ be a system of convex $\mathcal{L}$-subgroups of $G$. The lattice ordered group $G$ is said to be a completely subdirect product of the system $S$ if $
sum_{i \in I} G_i \subseteq G \subseteq \nprod_{i \in I} G_i$. (Cf. also Conrad [2].) (Recall that the symbol $\sum_{i \in I} G_i$ above denotes the restricted direct product of the system $S$.)

It is easy to verify that for $G \in G$ and $S = \{G_i\}_{i \in I} \subseteq \mathcal{C}(G)$ the following conditions are equivalent:

(i) $G$ is a completely subdirect product of the system $S$;

(ii) $G_i \cap G_j = \{0\}$ whenever $i$ and $j$ are distinct elements of $I$, and for each $0 < g \leq G$ there are $g_i \in G_i (i \in I)$ with $g = \bigvee_{i \in I} g_i$.

(iii) $G_i \cap G_j = \{0\}$ whenever $i$ and $j$ are distinct elements of $I$, each $G_i$ is a direct factor of $G$ and $\bigvee_{i \in I} G_i = G$.

3.3. Lemma. Let $G \in G$ and $H \in \mathcal{C}(G)$. Assume that $H$ is a completely subdirect product of a system $S = \{H_i\}_{i \in I}$, where each $H_i$ is linearly ordered. Then $H^\sim$ is a completely subdirect product of the system $S$ as well.

Proof. Without loss of generality we may suppose that $H_i \neq \{0\}$ for each $i \in I$. Let $0 < g \leq H^\sim$. Then $g$ is a join of some positive elements of $H_i$; since each positive element of $H$ is a join of positive elements belonging to $\bigcup_{i \in I} H_i$ there are elements $0 < h_j \leq H_i$ such that $g = \bigvee h_j$.

Let $i_1$ be a fixed element of $I$. Assume that $x \leq g$ for each $x \in H_{i_1}$. Choose $0 < y \in 154$
\[ \text{If } i \in I \text{ and } i \neq i', 0 < z \in H_i, \text{ then } z + y = z \vee y; \text{ hence} \]
\[ g < g + y = \bigvee h_j + y = \bigvee (h_j + y) \leq g, \]
which is a contradiction. Therefore for each \( i \in I \) there is \( 0 < h'_i \in H_i \) such that \( h'_i \leq g \). Then \( h'_i \wedge g \) is the greatest element of the set \([0, g] \cap H_j\). Hence we infer that \( g = \bigvee_{i \in I} h'_i \). Consequently, \( H^- \) is a completely subdirect product of the system \( S \).

3.4. Proposition. (Cf. [5], Thm. 3.4.) Let \( X \subseteq \mathcal{G} \). Assume that each lattice ordered group belonging to \( X \) is linearly ordered. Let \( G \in \mathcal{G} \). Then the following conditions are equivalent:

(i) \( G \in T(X) \).
(ii) There are systems \( \{A_i\}_{i \in I} \subseteq c(G) \) and \( \{A_{ij}\}_{j \in J(i)} \subseteq c(G) \cap X \) for each \( i \in I \), such that \( A_i = \bigcup_{j \in J(i)} A_{ij} \) is valid for each \( i \in I \), and \( G = \bigvee_{i \in I} A_i \).

For \( X \subseteq \mathcal{G} \) we denote by \( X^0 \) the class of all \( G \in \mathcal{G} \) having the property that there exists a linearly ordered system \( \{G_{ij}\}_{i \in I} \subseteq X \cap c(G) \) such that \( G = \bigvee_{i \in I} G_i \) is valid. (The system under consideration is ordered by inclusion.)

3.5. Theorem. Let \( X \subseteq \mathcal{G} \), \( X_1 = \text{Lat Sub} \ X \). Assume that all elements of \( X \) are linearly ordered groups. Then

(i) all elements of \( X_1 \) are linearly ordered groups;
(ii) \( T_K(X) \) is the class of all lattice ordered groups which can be expressed as completely subdirect products of linearly ordered groups belonging to \( X_1 \).

Proof. (i) is a consequence of 3.2. In view of 2.2 we have \( \text{Sub} \ X_1 = X_1 \), and clearly \( \text{Lat} \ X_1 = X_1 \). Thus 3.1, 3.3 and 3.4 imply that (ii) is valid.

3.6. Corollary. Let \( Y \) be a \( K \)-radical class. Then the following conditions are equivalent:

(i) Each \( G \in Y \) is a completely subdirect product of linearly ordered groups.
(ii) There exists a class \( X \) of linearly ordered groups such that \( Y = T_K(X) \).

3.7. Proposition. Let \( G, G' \in \mathcal{G} \). Assume that (i) \( G \) is a completely subdirect product of linearly ordered groups \( G_i \ (i \in I) \), and (ii) \( K(G) \) is isomorphic to \( K(G') \). Then \( G' \) can be expressed as a completely subdirect product of linearly ordered groups \( G'_i \ (i \in I) \) such that, for each \( i \in I \), \( K(G_i) \) is isomorphic to \( K(G'_i) \).

Proof. Let \( \varphi \) be an isomorphism of the lattice \( K(G) \) onto \( K(G') \). Put \( G'_i = \varphi(G_i) \) for each \( i \in I \). According to 2.6, the lattices \( K(G_i) \) and \( K(G'_i) \) are isomorphic. Thus in view of 3.2, all \( G'_i \) are linearly ordered.

\( G_i \) is a direct factor of \( G \) and hence \( G_i \) has a complement in the lattice \( K(G) \). Thus \( G'_i \) has a complement \( H'_i \) in the lattice \( K(G') \). Without loss of generality we may suppose that all the lattice ordered groups \( G_i \) are non-zero; hence the same is valid for \( G'_i \). Assume that there exists \( g' \in G' \) such that \( g' \) is an upper bound for some \( G'_{i_1} \ (i_1 \in I) \). We have
\[ G' = (G'_{i_1} \wedge H'_{i_1})^-, \]
hence \( g' \) is a join of some positive elements of \( G_i \lor H_i \). Each positive element of \( G_i \lor H_i \) is a join of some positive elements of the set \( G_i \cup H_i \); hence there are elements \( 0 < g_j \in G_i \cup H_i \) with \( g' = \bigvee g_j \). Choose \( 0 < k \in G_i \). Then

\[
g' < g' + k = \bigvee (g_j + k).
\]

If \( g_j \in G_i \), then \( g_j + k \in G_i \) and hence \( g_j + k \leq g' \); if \( g_j \in H_i \), then \( g_j + k = g_j \lor k \leq g' \). Thus \( \bigvee (g_j + k) \leq g' \), which is a contradiction. Therefore \( G_i \) fails to be bounded in \( G' \) and hence, in view of the result of [6], \( G_i \) is a direct factor in \( G' \).

We have \( \bigvee_{i \in I} G_i = G \), hence \( \bigvee_{i \in I} G_i = G' \). Therefore \( G' \) is a completely subdirect product of the system \( \{G_i\} \ (i \in I) \).

Remark. Proposition 3.7 can be applied to obtain an alternative proof of Theorem 3.5.

3.8. Example. Let \( R \) be the additive group of all reals with the natural linear order. We denote by \( \circ \) the operation of lexicographic product. Put \( R_n = R \circ R \circ \ldots \circ R \ (n\text{-times}), \ X_n = T_R(R_n) \ (n = 1, 2, \ldots), \)

\[
G = \prod R_n \ (n = 1, 2, \ldots),
\]

\[
X = \bigvee X_n, \quad Y = \bigvee X_n \ (n = 1, 2, \ldots).
\]

Then 3.4 and 1.2 imply that \( G \) does not belong to \( X \); on the other hand, according to 3.5 we have \( G \in Y \). Hence if \( S \) is an infinite chain in the lattice \( \mathcal{L}_{K} \), the join of \( S \) with respect to \( \mathcal{L}_{K} \) need not coincide with the join of \( S \) with respect to \( \mathcal{R} \). In particular, \( \mathcal{R}_{K} \) fails to be a closed sublattice of \( \mathcal{R} \).

4. ON ATOMS OF THE LATTICE \( \mathcal{R}_{K} \)

For \( X \in \mathcal{R}_{K} \) we denote by \( a(X) \) the class of all \( Y \in \mathcal{R}_{K} \) such that (i) \( X < Y \), and (ii) there exists no \( Z \in \mathcal{R}_{K} \) with \( X < Z < Y \). The elements of \( a(X) \) are called atoms over \( X \). If \( X = \{\{0\}\} = R_0 \), then \( a(X) \) is the class of all atoms of \( \mathcal{R}_{K} \); this class will be denoted by \( A_0 \). Also, for each \( X \in \mathcal{R}_{K} \), we denote by \( a_1(X) \) the class of all \( Y \in \mathcal{R}_{K} \) such that (i) \( X < Y \), and (ii) the interval \( [X, Y] \) of \( \mathcal{R}_{K} \) has no atoms. The following proposition shows that \( A_0 \) is nonempty.

4.1. Proposition. Let \( X \in \mathcal{R}_{K} \). Then the following conditions are equivalent:

(i) \( X \in A_0 \) and \( X \) contains an archimedean non-zero linearly ordered group.

(ii) \( X \) is the class of all lattice ordered groups which can be expressed as completely subdirect products of archimedean linearly ordered groups.

Proof. Let us denote by \( Y \) the class of all lattice ordered groups \( G \) such that \( G \) is a completely subdirect product of archimedean linearly ordered groups. In view of 3.5, \( Y \) is a \( K \)-radical class. Let \( Z \in \mathcal{R}_{K}, R_0 < Z \leq Y \). There is \( G_1 \in Z \) with \( G_1 \neq \{0\} \);
further, there is an archimedean linearly ordered group \( G_2 \neq \{0\} \) such that \( G_2 \) is a direct factor of \( G \). If \( G_3 \) is any non-zero archimedean linearly ordered group, then \( K(G_3) \) is isomorphic to \( K(G_2) \), hence \( G_3 \in Z \) and therefore (because \( Z = \text{Join } Z \)) we infer that \( Y \subseteq Z \). Thus (i) is a consequence of (ii). The implication (i) \( \Rightarrow \) (ii) is obvious.

The word “archimedean” cannot be omitted in the condition (i) of 4.1. This will be shown by the following construction and by 4.2.

Let \( \alpha \) be an infinite cardinal. We denote by \( \omega(\alpha) \) the first ordinal having the property that the set of all ordinals less then \( \omega(\alpha) \) has the cardinality \( \alpha \). Let \( I_\alpha \) be a linearly ordered set dual to \( \omega(\alpha) \). Let \( G \) be a linearly ordered group, \( G \neq \{0\} \). We put

\[
G_\alpha = \Gamma_{\alpha I(\alpha)} G_i,
\]

where \( \Gamma \) is the symbol denoting the lexicographic product and \( G_i \) is isomorphic to \( G \) for each \( i \in I \).

From the definition of \( G_\alpha \) we immediately obtain:

4.2. Lemma. Let \( H \) be a convex \( l \)-subgroup of \( G_\alpha, \ H \neq \{0\} \). Then there exists a convex \( l \)-subgroup \( H_1 \) of \( H \) such that \( H_1 \) is isomorphic to \( G_\alpha \).

4.3. Lemma. Let \( \{0\} \neq G \in \mathcal{B} \). Assume that for each \( H \in c(G) \) there exists \( H_1 \in c(H) \) such that \( H_1 \) is isomorphic to \( G \). Then \( T_k(G) \) is an atom in \( \mathcal{R}_k \).

Proof. We have \( R_0 < T_k(G) \). Let \( X \in \mathcal{R}_k, \ R_0 < X \leq T_k(G) \). There exists \( \{0\} \neq G' \in X \). According to 2.9 we have \( G' \in \text{Join } \text{Lat } \text{Sub } \{G\} \). Hence there are \( \{G_i\}_{i \in I} \subseteq \subseteq K(G'), \{H_i\}_{i \in I} \subseteq \text{Sub } \{G\} \) such that \( G' = \bigvee_{i \in I} G_i \) and for each \( i \in I \) there exists an isomorphism \( \varphi_i \) of \( K(G_i) \) onto \( K(H_i) \). Without loss of generality we can suppose that \( G_i \neq \{0\} \) for each \( i \in I \); hence \( H_i \neq \{0\} \) for each \( i \in I \). In view of 2.2, \( K(H_i) \) is isomorphic to \( K(H_i^-) \), hence there is an isomorphism \( \varphi_i^- \) of \( K(G_i) \) onto \( K(H_i^-) \). According to the assumption there exists \( H_{1i} \in c(H_i^-) \) such that \( H_{1i} \) is isomorphic to \( G \). Hence \( K(H_{1i}) \) is isomorphic to \( K(G) \) and in view of 2.2, \( K(H_{1i}) \) is isomorphic to \( K(G) \) as well. Thus \( (\varphi_i^-)^{-1} (H_{1i}) \in T_k(G') \subseteq X \) and \( K((\varphi_i^-)^{-1} (H_{1i})) \) is isomorphic to \( K(G) \), implying \( G \in T_k(G') \). Therefore \( X = T_k(G) \).

4.4. Proposition. Let \( G \in \mathcal{B}, \ G \neq \{0\} \) and let \( \alpha \) be an infinite cardinal. Then \( T_k(G_\alpha) \) is an atom of the lattice \( \mathcal{R}_k \).

Proof. This is a consequence of 4.2 and 4.3.

4.5. Corollary. The class of all atoms of \( \mathcal{R}_k \) is a proper class.

Let us remark that \( T_k(G_\alpha) \) does not contain any nonzero archimedean \( l \)-group.

4.6. Proposition. There exists a nonzero linearly ordered group \( G \) such that the interval \( [R_\ell, T_k(G)] \) of the lattice \( \mathcal{R}_k \) contains no atom.

Proof. Let \( \beta \) be an infinite cardinal and let \( l(\beta) \) be a linearly ordered set dual...
to \(\omega(\beta)\). For each \(i \in I(\beta)\) let \(\alpha_i\) be a cardinal such that
\[
i_1 > i_2 \Rightarrow \alpha_{i_1} < \alpha_{i_2};
\]
further, let \(J_i\) be a linearly ordered set isomorphic to \(\omega(\alpha_i)\). We denote by \(M\) the set
of all pairs \(m = (j, i)\) with \(i \in I(\beta)\) and \(j \in J_i\); we put \((j_1, i_1) < (j_2, i_2)\) if either
\(i_1 < i_2\), or \(i_1 = i_2\) and \(j_1 < j_2\). Let \(G_0 \neq \{0\}\) be an archimedean linearly ordered
group and for each \(m \in M\) let \(G_m\) be a linearly ordered group isomorphic to \(G_0\). Put
\[
G = \Gamma_{m \in M} G_m.
\]

Let \(R_0 < X \leq T_k(G)\). There exists \(H_1 \in X\) with \(H_1 \neq \{0\}\). In view of 3.5, there
exists a convex \(l\)-subgroup \(H_2 \neq \{0\}\) of \(G\) such that \(H_2\) is isomorphic to a convex
subgroup of \(H_1\); thus \(T_k(H_2) \leq X\). The construction of \(G\) yields that there exists
a convex \(l\)-subgroup \(H_3 \neq \{0\}\) of \(H_2\) such that for each nonzero convex \(l\)-subgroup
\(H_4\) of \(H_3\) the lattice \(K(H_4)\) fails to be isomorphic to \(K(H_2)\). By using 3.5 again we
infer that \(H_2\) does not belong to \(T_k(H_3);\) therefore \(R_0 < T_k(H_3) < T_k(H_2)\). Thus \(X\)
fails to be an atom of \(\mathcal{R}_k\).

We have proved that the class \(a_1(R_0)\) is nonempty. This result can be sharpened
as follows. Let us apply the same notation as in the proof of 4.6.

Now if we choose a cardinal \(\beta_1\) such that \(\beta_1 > \beta\) and if we construct a linearly
ordered group \(G_1\) in the same way as we did for \(G\) (with the distinction that \(\beta\) is
replaced by \(\beta_1\)), then in view of 3.5, \(G_1\) does not belong to \(T_k(G);\) therefore \(T_k(G_1) \neq \{0\}\)
\(\neq T_k(G)\). From this consideration we infer

4.7. Proposition. \(a_1(R_0)\) is a proper class.

Denote \(A = \sup a(R_0)\), \(A_1 = \sup a_1(R_0)\) (the symbol \(\sup\) being taken with respect
to the complete lattice \(\mathcal{R}_k\)).

The following proposition is a consequence of 4.5, 4.7 and of the fact that \(\mathcal{R}_k\) is
a Brouwer lattice (cf. 2.11).

4.8. Proposition. \(a(A_1)\) and \(a_1(A)\) are proper classes. The class \(a_1(A_1)\) is empty.
The lattice \([R_0, A]\) is complemented. \(A_1\) belongs to \(a_1(R_0)\).

5. THE CLASS \(a(A)\)

Let \(G \in \mathcal{G}, H \in e(G)\). If for each \(g \in G \setminus H\) with \(g > 0\) we have \(g > h\) whenever
\(h \in H\), then \(G\) is said to be a lexicographic extension of the lattice ordered group \(H\) and we
denote this fact by writing \(G = \langle H \rangle\). The lattice ordered group \(G\) is called a proper
lexico extension if there exists \(H \in e(H)\) with \(G = H \neq \{0\}\) such that \(G = \langle H \rangle\).

Let \(G \neq \{0\}\) be an archimedean linearly ordered group. Let \(\alpha\) be an infinite car­
dinal. Put
\[
G_{[\alpha]} = \left(\prod_{i \in I} G_i\right) \cdot G,
\]
where each \(G_i\) is isomorphic to \(G\). Then \(G_{[\alpha]}\) is a proper lexicographic extension.
5.1. **Lemma.** $T_k(G)$ is covered by $T_k(G_{[a]}).$

**Proof.** There exists a convex $l$-subgroup of $G_{[a]}$ isomorphic to $G$; thus $G \in T_k(G_{[a]})$ and hence $T_k(G) \leq T_k(G_{[a]})$. Now 3.4 implies that $G_{[a]}$ does not belong to $T_k(G)$ (in fact, each element of $T_k(G)$ is archimedean and $G_{[a]}$ fails to be archimedean). Therefore $T_k(G) < T_k(G_{[a]}).

Let $X \in \mathcal{K}_k$, $T_k(G) < X \leq T_k(G_{[a]})$. Hence there is $H \in X \setminus T_k(G)$. In view of 2.9 there are $H_i \in K(H)$ ($i \in I$) such that $H = \bigvee_{i \in I} H_i$ and each $H_i$ belongs to Lat Sub $\{G_{[a]}\}$. Hence for each $i \in I$ there is $H'_i \in \text{Sub} \{G_{[a]}\}$ such that $K(H_i)$ is isomorphic to $K(H'_i)$.

For each $H'_i$ we have either (i) $H'_i \subseteq \prod_{i \in I} G_i$, or (ii) $H'_i = G_{[a]}$. If (i) is valid for each $i \in I$, then $H'_i \in T_k(G)$ holds for each $i \in I$, hence $H \in T_k(G)$, which is a contradiction. Therefore there is $i \in I$ fulfilling (ii) and so $X = T_k(G_{[a]})$, completing the proof.

5.2. **Lemma.** Let $X \in A_0$, $X \neq T_k(G)$. Then $T_k(G_{[a]}) \land X = R_0$.

**Proof.** By way of contradiction, assume that there is $H \in T_k(G_{[a]}) \land X$ with $H \neq \{0\}$. The construction of $G_{[a]}$ and 2.9 yields that each nonzero lattice ordered group belonging to $T_k(G_{[a]})$ has a nonzero archimedean subgroup. Thus $T_k(G) \leq X$, which is a contradiction.

5.3. **Lemma.** $T_k(G_{[a]}) \lor A$ covers $A$.

**Proof.** Let $A'_0$ be the class of all elements of $A_0$ distinct from $T_k(G)$ and let $A' = \sup A'_0$. In view of 5.2 and 2.11 we have

$$T_k(G_{[a]}) \land A' = R_0,$$

hence according to 5.1 we obtain

$$T_k(G_{[a]}) \land A = T_k(G_{[a]}) \land (T_k(G) \lor A') = T_k(G).$$

Since the interval $[A, T_k(G_{[a]}) \lor A]$ is transposed to the interval $[T_k(G), T_k(G_{[a]})]$, it follows from 5.1 that $T_k(G_{[a]}) \lor A$ covers $A$.

5.4. **Lemma.** Let $\beta$ be a cardinal, $\beta > \alpha$. Then $T_k(G_{[a]}) \neq T_k(G_{[\beta]}).$

**Proof.** By way of contradiction, assume that $T_k(G_{[a]}) = T_k(G_{[\beta]})$. Hence $G_{[\beta]} \in T_k(G_{[a]})$. In view of 2.9 there are $H_j \in K(G_{[\beta]})$ ($j \in J$) such that $G_{[\beta]} = \bigvee_{j \in J} H_j$ and all $H_j$ belong to Lat Sub $\{G_{[a]}\}$. Let $j \in J$. There exists $H'_j \in \text{Sub} \{G_{[a]}\}$ such that $K(H_j)$ is isomorphic to $K(H'_j)$. We distinguish two cases.

a) $H'_j \subseteq \prod_{i \in I} G_i$ for each $j \in J$. Because $\prod_{i \in I} G_i = \bigvee_{i \in I} G_i$ and each $G_i$ is isomorphic to $G$, we infer that $G_{[\beta]}$ belongs to $T_k(G)$ which is a contradiction (in fact, $G$ is linearly ordered and archimedean, hence in view of 3.4 all elements of $T_k(G)$ are archimedean, but $G_{[\beta]}$ fails to be archimedean).

b) There exists $j \in J$ such that $H'_j \nsubseteq \prod_{i \in I} G_i$. Then $H'_j = G_{[a]}$. Thus $K(H_j)$ cannot be isomorphic to $K(H'_j)$, and we arrived at a contradiction.
5.5. Corollary. Let $\beta$ be as in 5.4. Then

\[(T_K(G_{[a]}) \lor A) \land (T_K(G_{[\beta]} \lor A) = A.\]

5.6. Theorem. $a(A)$ is a proper class.

Proof. For each infinite cardinal $\alpha$ we have $T_K(G_{[a]}) \lor A \in a(A)$ (cf. 5.3). From 5.5 it follows that if $\beta > \alpha$, then $T_K(G_{[a]} \lor A \neq T_K(G_{[\beta]} \lor A$. Hence $a(A)$ is a proper class.

6. PRINCIPAL ELEMENTS OF $\mathcal{R}_K$

An element $X \in \mathcal{R}_K$ is said to be principal if there is $G \in \mathcal{G}$ such that $X = T_K(G)$. Let $\mathcal{P}$ be the class of all principal elements of $\mathcal{R}_K$.

To each $G \in \mathcal{G}$ we assign a cardinal $k(G)$ as follows: $k(G)$ is the least cardinal $\alpha$ having the property that there are $G_i \in K(G)$ ($i \in I$) such that $\bigvee_{i \in I} G_i = G$ and $\text{card} K(G_i) \leq \alpha$ for each $i \in I$.

A subclass $X$ of $G$ is said to be $k$-bounded if there is a cardinal $\alpha$ such that $k(G) \leq \alpha$ for each $G \in X$.

6.1. Proposition. Let $X \in \mathcal{R}_K$. Then the following conditions are equivalent:

(i) $X$ is a principal element of $\mathcal{R}_K$.

(ii) $X$ is $k$-bounded.

Proof. Suppose that $X$ is principal, $X = T_K(G)$. Put $\text{card} K(G) = \alpha$. Then 2.9 implies that for each $H \in X$ we have $k(H) \leq \alpha$; hence $X$ is $k$-bounded.

Conversely, assume that $X$ is $k$-bounded, i.e., $k(H) \leq \alpha$ for each $H \in X$. There exists a set $S$ of lattices which has the following properties:

(i) If $L \in S$, then $\text{card} L \leq \alpha$.

(ii) If $L \in S$, then there is $H_1 \in X$ such that $L$ is isomorphic to $K(H_1)$.

(iii) If $H_1 \in X$ and $\text{card} K(H_1) \leq \alpha$, then there is $L \in S$ such that $L$ is isomorphic to $K(H_1)$.

For each $L \in S$ we choose a fixed $H_1$ fulfilling (ii) and we put $H_1 = H_1(L)$. We denote

\[G = \prod_{L \in S} H_1(L).\]

Because $H_1(L) \in X$ and since $X$ is closed with respect to direct products (cf. [1]) we infer that $G$ belongs to $X$ and hence $T_K(G) \subseteq X$. Let $H \in X$. In view of the definition of $S$ there are $H_i$ ($i \in I$) in $K(G)$ such that $\bigvee_{i \in I} H_i = H$ and for each $i \in I$, $K(H_i)$ is isomorphic to a certain element of $S$. Hence $H_i \in T_K(G)$ for each $i \in I$ and thus $H \in T_K(G)$. Therefore $X = T_K(G)$.

6.2. Corollary. Let $X \in \mathcal{P}$, $Y \in \mathcal{R}_K$, $Y \subseteq X$. Then $Y \in \mathcal{P}$.

From 6.1 and 2.10 we also obtain:
6.3. Corollary. Let $I$ be a nonempty set and for each $i \in I$ let $X_i$ be a principal $K$-radical class. Then $\bigvee_{i \in I} X_i$ is a principal $K$-radical class as well.

6.4. Proposition. Let $A$ and $A_1$ be as in Sec. 4. Then $A \notin \mathcal{P}$ and $A_1 \notin \mathcal{P}$.

Proof. Let $G \in \mathcal{G}$, $G \neq \{0\}$ and let $\alpha$ be an infinite cardinal. Then $k(G_\alpha) \geq \alpha$. Hence $\{k(H)\}_{H \in A}$ is a proper class of cardinals. Thus the class $A$ fails to be $k$-bounded and in view of 6.1, $A$ is not principal. Also, from the construction of elements of $A_1$ applied in the proof of 4.6 it follows that $A_1$ is not $k$-bounded, hence $A_1$ fails to be principal.

6.5. Theorem. Let $X$ be a principal $K$-radical class. Then both $a(X)$ and $a(X^1)$ are proper classes.

Proof. According to 6.1 there is a cardinal $\alpha$ such that $k(G) < \alpha$ for each $G \in X$. Let $G' \in \mathcal{G}$, $G' \neq \{0\}$. For each $\beta > \alpha$ we have $k(G'_\beta) > \alpha$, hence $G'_\beta \notin X$ and thus, because of 4.4, $X \cap T_k(G'_\beta) = R_0$ and thus $X$ is covered by $X \vee T_k(G'_\beta)$. If $\beta_1 > \beta$, then (since $\mathcal{R}_K$ is a Brouwer lattice) we have $X \vee T_k(G'_\beta) \neq X \vee T_k(G'_\beta)$. Therefore $a(X)$ is a proper class. The proof for $a(X^1)$ is analogous.

7. $K$-Radical Classes Having No Cover

In view of the above results the natural question arises: does there exist a $K$-radical class $X \neq \emptyset$ having the property that $a(X) = \emptyset$? The following consideration (the idea of which is analogous to that performed in [4] for the lattice $\mathcal{R}$ of all radical classes) shows that the answer is affirmative.

7.1. Lemma. Let $X, Y \in \mathcal{R}_K$ such that $X$ is covered by $Y$. Then there are $G_1 \in Y \setminus X$, $G_2 \in X$ such that $T_k(G_2)$ is covered by $T_k(G_1)$.

Proof. From $X < Y$ it follows that there is $G_1 \in Y \setminus X$. Hence $T_k(G_1) \leq Y$ and $T_k(G_1) \notin Z \in X$. Put $T_k(G_1) \cap X = Z$. In view of 6.2, $Z$ is a principal $K$-radical class, i.e., there is $G_2 \in Z$ with $Z = T_k(G_2)$. Then $G_2 \in X$. Since $X$ is covered by $Y$ we have $X \vee T_k(G_1) = Y$. Since the interval $[T_k(G_2), T_k(G_1)]$ is transposed to $[X, Y]$, $T_k(G_2)$ is covered by $T_k(G_1)$.

7.2. Corollary. Let $X \in \mathcal{R}_K$. Assume that for each $G_2 \in X$ and each $G_1 \in \mathcal{G}$ such that $T_k(G_1) \in a(T_k(G_2))$ we have $G_1 \in X$. Then $a(X) = \emptyset$.

Let $A_1$ be as in Sec. 4.

7.3. Lemma. Let $X \in \mathcal{R}_K$, $X \wedge A_1 = R_0$ and $Y \in a(X)$. Then $Y \wedge A_1 = R_0$.

Proof. By way of contradiction, assume that $Y \wedge A_1 = Z > R_0$. According to 4.8, the interval $[R_0, Z]$ contains no atom. Because $X$ is covered by $Y$ and $Z \notin X$, we obtain $X \vee Z = Y$. Since $X \wedge Z = R_0$, the interval $[R_0, Z]$ is transposed to $[X, Y]$, hence $R_0$ is covered by $Z$, which is a contradiction.
For each \( C \subseteq \mathscr{R}_k \) we denote by \( J(C) \) the class of all \( Y \in \mathscr{P} \) such that there are \( Z_1 \in C \) and \( G_1 \in Z_1 \) with \( Y \in a(T_k(G_1)) \). Next we put \( X_1 = \sup \cup \cup \) \( J(C) \).

For each ordinal \( \alpha \) we assign a \( K \)-radical class \( X_\alpha \) as follows. We put \( X_1 = R_0 \).

Suppose that we have defined \( X_\beta \) for each \( \beta < \alpha \). If \( \alpha \) is a limit ordinal, then we put \( X_\alpha = \bigvee_{\beta < \alpha} X_\beta \).

If \( \alpha \) is non-limit, \( \alpha = \gamma + 1 \), then we set \( X_\alpha = X_\gamma \left( [R_0, X_\gamma] \right) \).

Put \( X_0 = \bigvee_\alpha X_\alpha \), where \( \alpha \) runs over the class of all ordinals.

7.4. Proposition. \( a(X_0) = 0 \) and \( X_0 \neq \emptyset \).

Proof. The relation \( a(X_0) = 0 \) follows from the construction of \( X_0 \) and from 7.2. According to 7.3 we have \( X_0 \land A_1 = R_0 \), hence \( X_0 \neq \emptyset \).

Let us remark that if \( G \in X \in \mathscr{R}_K, G' \in \mathscr{P} \setminus X, T_k(G') \in a(T_k(G)) \), then \( X \lor T_k(G') \in \in \in \in a(X) \). By applying this remark and using a construction analogous to that of \( X_0 \) the following proposition can be proved (the detailed proof will be omitted):

7.5. Proposition. Let \( X \in \mathscr{R}_K \). There exists \( Y \in \mathscr{R}_K \) such that

(1) \( X \leq Y \);

(2) \( a(X) = 0 \);

(3) if \( Z \in \mathscr{R}_K \), \( X \leq Z \) and \( a(Z) = 0 \), then \( Y \leq Z \).

Let \( X_A, X_B, X_C \) and \( S_i \) be as in Introduction. Now we shall prove that \( a(X) \neq 0 \) for each \( X \in \{ X_A, X_B, X_C, S_i \} \).

7.6. Proposition. \( a(X_A) \) is a proper class.

Proof. Let \( G \) be a nonzero archimedean linearly ordered group. For each infinite cardinal \( \alpha \) let \( G(\alpha) \) be as in Sec. 5. Put \( Y_\alpha = X_A \lor T_k(G(\alpha)) \). We have \( G(\alpha) \notin X_A \). Hence in view of 5.1 we infer that \( Y_\alpha \in a(X_A) \). Now 5.4 implies that \( a(X_A) \) is a proper class.

Let us denote by \( F \) the additive group of all real functions \( f \) defined on the set \( M \) of all rational numbers \( x \in (0, 1) \) which have the following property: there are irrational numbers \( x_1, \ldots, x_n \in (0, 1) \) with \( x_1 < x_2 < \ldots < x_n \) (depending on \( f \)) such that \( f \) is a constant on each of the sets \( (0, x_1) \cap M, (x_1, x_2) \cap M, \ldots, (x_n, 1) \cap M \).

For \( f_1, f_2 \in F \) we put \( f_1 \leq f_2 \) if \( f_1(x) \leq f_2(x) \) is valid for each \( x \in M \). Then \( F \) is a lattice ordered group.

It follows immediately from the construction of \( F \) that each nonzero convex \( l \)-subgroup of \( F \) is isomorphic to \( F \). Hence in view of 4.3 we obtain:

7.7. Lemma. \( T_k(F) \) is an atom of \( \mathscr{R}_K \).

7.8. Proposition. \( a(X_B) \neq 0 \).

Proof. If \( \{0\} \neq G \in X_B \), then there is a nonzero convex \( l \)-subgroup \( H \) of \( G \) such that \( K(H) \) is a chain (cf. 3.2). If \( G \in c(F) \), then \( G \) does not have the mentioned property; hence according to 2.9 we have \( X_B \land T_k(F) = R_0 \). Therefore in view of 7.7, \( X_B \lor T_k(G) \) covers \( X_B \).

Also, no nonzero convex \( l \)-subgroup of \( F \) is completely distributive; hence by analogous reasoning as in the proof of 7.8 we infer:
7.9. Proposition. \( a(X_c) = \emptyset \).

7.10. Lemma. (Cf. [1], p. 191.) Let \( G \in \mathcal{S} \). Then \( G \) belongs to \( S \) if and only if each polar of \( G \) is complemented in \( K(G) \).

Let \( G_{[z]} \) be as above. Then 7.10 implies that \( G_{[z]} \) does not belong to \( S \).

7.11. Lemma. \( T_k(G_{[z]}) \wedge S = T_k(G) \).

Proof. We have \( T_k(G) < T_k(G_{[z]}) \) and in view of 7.10, \( T_k(G) \leq S \); hence we have to verify that \( T_k(G_{[z]}) \wedge S \leq T_k(G) \).

Let \( H \in T_k(G_{[z]}) \wedge S \). Then there are \( H_j (j \in J) \) in \( K(H) \) \( \cap \) \( \text{Lat Sub } \{ G_{[z]} \} \) such that \( H = \bigvee_{j \in J} H_j \). Hence there exist \( H'_j \in \text{Sub } \{ G_{[z]} \} \) such that \( K(H'_j) \) is isomorphic to \( K(H_j) \) for each \( j \in J \). Because of \( H \in S \), we have \( H_j \in S \) and \( H'_j \in S \) for each \( j \in J \). Thus \( H'_j \neq G_{[z]} \) and therefore \( H'_j \leq \prod_{i \in J} G_i \) for each \( j \in J \) (the denotations are as in Sec. 5). Hence \( H_j \in T_k(G) \) and so \( H_j \in T_k(G) \) for each \( j \in J \), implying \( H \in T_k(G) \).

7.12. Proposition. \( a(S_i) \) is a proper class.

Proof. From 7.11 and 5.1 we infer that \( T_k(G_{[z]}) \vee S \) belongs to \( a(S_i) \). Therefore in view of 5.4, \( a(S_i) \) is a proper class.

References


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