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ON $k$-DOMATIC NUMBERS OF GRAPHS

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In [1] M. Borowiecki and M. Kuzak have generalized the concept of a dominating set in a graph. Let $G$ be an undirected graph without loops and multiple edges, let $k$ be a positive integer. A $k$-dominating set in the graph $G$ is a subset $D$ of the vertex set $V(G)$ of $G$ with the property that for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ such that $d(x, y) \leq k$. (The symbol $d(x, y)$ denotes the distance of the vertices $x, y$ in the graph $G$.) For $k = 1$ the $k$-dominating sets are dominating sets in the usual sense.

This leads to a generalization of the concept of the domatic number of a graph which was introduced by E. J. Cockayne and S. T. Hedetniemi in [2]. A $k$-domatic partition of $G$ is a partition of $V(G)$, all of whose classes are $k$-dominating sets in $G$. The maximum number of classes of a $k$-domatic partition of $G$ is called the $k$-domatic number of $G$ and denoted by $d_k(G)$.

For $k = 1$ we have $d_k(G) = d(G)$, where $d(G)$ is the domatic number of $G$.

**Proposition 1.** Let $k, l$ be positive integers, $k < l$. Let $G$ be an undirected graph. Then $d_k(G) \leq d_l(G)$.

**Proof.** From the definition of a $k$-dominating set it is clear that each $k$-dominating set in $G$ is also $l$-dominating in $G$ and hence each $k$-domatic partition of $G$ is an $l$-domatic partition of $G$. This implies the assertion. □

**Proposition 2.** Let $G$ be an undirected graph with $n$ vertices, let $D(G)$ be its diameter. Then $d_k(G) = n$ for each $k \geq D(G)$.

**Proof.** Let $k \geq D(G)$, let $x \in V(G)$. For each $y \in V(G)$ we have $d(x, y) \leq D(G) \leq k$, therefore $\{x\}$ is a $k$-dominating set in $G$. The partition of $V(G)$ into one-element sets is a $k$-domatic partition of $G$; it has $n$ classes and no partition of $V(G)$ can have more than $n$ classes. This implies $d_k(G) = n$. □

**Proposition 3.** Let $G$ be an undirected graph, let $G'$ be its spanning subgraph. Then $d_k(G) \geq d_k(G')$.

**Proof.** The assertion follows from the fact that $V(G') = V(G)$ and the distance of arbitrary two vertices in $G'$ is greater than or equal to that in $G$. □
Proposition 4. Let $G$ be an undirected graph, let $k$ be a positive integer. Then $d_k(G)$ is equal to the minimum of $k$-domatic numbers of all connected components of $G$.

The proof is left to the reader.

Theorem 1. Let $G$ be a connected undirected graph with $n$ vertices, let $k$ be a positive integer. Then

$$d_k(G) \geq \min(n, k + 1).$$

Proof. If $n \leq k + 1$, then the diameter of $G$ is at most $k$, therefore $d_k(G) = n$. Suppose that $n > k + 1$. Choose a spanning tree $T$ of $G$. If the diameter of $T$ is less than or equal to $k$, then so is the diameter of $G$ and $d_k(G) = n$. If the diameter of $T$ is greater than $k$, let $c$ be a centre of $T$. Let $P$ be a diametral path in $T$; the vertex $c$ lies on $P$. Let $P_1, P_2$ be two subpaths of $P$ whose union is the whole $P$ and which have exactly one vertex in common, namely $c$. If $T$ has two centres, then we suppose (without loss of generality) that the centre different from $c$ lies on $P_1$. Let $B_1$ be the subtree of $T$ whose vertex set consists of all vertices $x$ with the property that $c$ does not lie between $x$ and any vertex of $P_1$. We shall colour the vertices of $T$ by the colours $0, 1, \ldots, k$ in the following way. The vertex $c$ is coloured by $0$. Any vertex of $B_1$ is coloured by the colour $i$ such that $i \in \{0, 1, \ldots, k\}$ and $i \equiv -d(c, x) \pmod{(k + 1)}$. Any vertex $x$ of $T$ not lying in $B_1$ is coloured by the colour $i$ such that $i \in \{0, 1, \ldots, k\}$ and $i \equiv d(c, x) \pmod{(k + 1)}$. In both these cases $d(c, x)$ denotes the distance of $c$ and $x$ in $T$. As the diameter of $T$ is greater than $k$, the path $P_1$ has a length at least $\lceil k/2 \rceil$ and contains the vertices of all the colours $\lceil k/2 \rceil + 1, \ldots, k$; the path $P_2$ has a length at least $\lceil k/2 \rceil$ and contains the vertices of all the colours $1, \ldots, \lceil k/2 \rceil$. (Here and in the sequel for an arbitrary real number $a$ the symbol $\lceil a \rceil$ denotes the greatest integer which is less than or equal to $a$ and the symbol $\lfloor a \rfloor$ denotes the least integer which is greater than or equal to $a$.) Let $D_i$ be the set of all vertices of $T$ which are coloured by the colour $i$ (for $i = 0, 1, \ldots, k$). Let $i$ be an arbitrary one from the numbers $0, 1, \ldots, k$; we shall prove that $D_i$ is a $k$-dominating set in $T$. Let $x \in V(T) - D_i$; then $x \in D_j$ for some $j$ distinct from $i$. Suppose $i < j$. If $x$ does not lie in $B_1$, then on the path connecting $x$ with $c$ there is a vertex $y$ such that $d(c, y) = d(c, x) - j + i$; we have $y \in D_i$ and $d(x, y) = j - i \leq k$. If $x$ lies in $B_1$ and $d(c, x) \geq k + 1$, then there exists a vertex $y$ in $B_1$ such that $d(c, y) = d(c, x) - k - 1 - j + i$; we have $y \in D_i$ and $d(x, y) = k + 1 + i - j \leq k$. If $x$ lies in $B_1$ and $d(c, x) \leq k$, then $d(c, x) = k + 1 - j$ and there exists a vertex $y$ on $P_2$ such that $d(c, y) = i$; we have $y \in D_i$ and $d(x, y) = k + 1 - j + i \leq k$. Now suppose $i > j$. If $x$ lies in $B_1$, then on the path connecting $x$ with $c$ there is a vertex $y$ such that $d(c, y) = d(c, x) - i + j$; we have $y \in D_j$ and $d(x, y) = i - j \leq k$. If $x$ does not lie in $B_1$ and $d(c, x) \leq k + 1$, then on the path connecting $x$ and $c$ there exists a vertex $y$ such that $d(c, y) = d(c, x) - k - 1 + i - j$; we have $y \in D_i$ and $d(x, y) = d(c, x) = k + 1 - i + j \leq k$. If $x$ does not lie in $B_1$ and $d(c, x) \leq k$, then $d(c, x) = j$.
and on $P_1$ there exists a vertex $y$ such that $d(c, y) = k + 1 - i$; then $y \in D_i$ and $d(x, y) = k + 1 - i + j \leq k$. We have proved that $D_i$ is a $k$-dominating set in $T$. As $i$ was chosen arbitrarily, $\{D_0, D_1, \ldots, D_k\}$ is a $k$-domatic partition of $T$ with $k + 1$ classes and $d_i(T) \geq k + 1$. According to Proposition 3 we have $d_k(G) \geq d_i(T) \geq k + 1$. $\square$

A graph consisting of one path will be called a snake.

**Theorem 2.** Let $G$ be a snake with $n$ vertices, let $k$ be a positive integer. Then

$$d_k(G) = \min (n, k + 1).$$

**Proof.** According to Theorem 1 the $k$-domatic number of $G$ is at least $\min (n, k + 1)$. If $n \leq k + 1$, it evidently cannot be greater. Thus suppose that $n > k + 1$. Let $u$ be a terminal vertex of $G$. There are exactly $k + 1$ vertices of $G$ whose distances from $u$ are at most $k$. If $\mathcal{P}$ is a partition of $V(G)$ into at least $k + 2$ classes, then at least one class of $\mathcal{P}$ contains none of these vertices. This class is not a $k$-dominating set in $G$, thus $\mathcal{P}$ is not a $k$-domatic partition of $G$. Hence $d_k(G) = k + 1 = \min (n, k + 1)$. $\square$

**Theorem 3.** Let $k, n$ be two positive integers, let $2 \leq k < n$. Then for each integer $m$ such that $k + 1 \leq m \leq n$ there exists a tree $T_m$ with $n$ vertices such that $d_k(T_m) = m$.

**Proof.** According to Theorem 2 a snake with $n$ vertices may be taken as $T_{k+1}$. Now let $k + 2 \leq m \leq n$. Let $a = \lceil n/m \rceil$. Take a snake $S$ with $a(k + 1)$ vertices. Let $u$ be a terminal vertex of $S$. Let $v$ be the vertex of $S$ adjacent to $u$. To each vertex of $S$ distinct from $v$ whose distance from $u$ is congruent with $1$ modulo $k + 1$ (there are exactly $a - 1$ such vertices) we add $m - k - 1$ new vertices and join them with it by edges. To $v$ we add $n - am + m - k - 1$ new vertices and join them with it by edges. We obtain a tree $T_m$ which has evidently $n$ vertices. Now we colour the vertices of $T_m$ by the colours $0, 1, \ldots, m - 1$. If $x$ is a vertex of $S$, then we colour it by the colour $i$ such that $i \in \{0, 1, \ldots, k\}$ and $i \equiv d(u, x) \pmod{(k + 1)}$. If $y$ is a vertex of $S$ such that $y \neq v$ and $d(u, y) \equiv 1 \pmod{(k + 1)}$, then to $y$ we have added $m - k - 1$ new vertices; we colour them by the colours $k + 1, \ldots, m - 1$. The vertices adjacent to $v$ and not belonging to $S$ will be coloured also by the colours $k + 1, \ldots, m - 1$; some of these colours may be repeated. (We have $n - am + m - k - 1 \geq m - k - 1$, because $a \leq n/m$.) Let $D_i$ be the set of all vertices of $T_m$ coloured by the colour $i$ (for $i = 0, 1, \ldots, m - 1$). We shall prove that each $D_i$ is a $k$-dominating set in $T_m$. First suppose $i \leq k$. Let $x \in V(T_m) - D_i$; then $x \in D_j$ for some $j \neq i$. If $j < i$, then $x$ belongs to $S$. If $d(u, x) \leq k$, then $d(u, x) = j$. There exists a vertex $y$ of $S$ such that $d(u, y) = i$; we have $y \in D_i$ and $d(x, y) = i - j \leq k$. If $d(u, x) \geq k + 1$, then there exists a vertex $y$ of $S$ such that $d(u, y) = d(u, x) - k + i - j - 1$; we have $y \in D_i$ and $d(x, y) = k - i + j + 1 \leq k$. If $i < j \leq k$, then $x$ belongs to $S$. There exists a vertex $y$ of $S$ such that $d(u, y) = d(u, x) + i - j$;
we have $y \in D_i$ and $d(x, y) = j - i \leq k$. If $j > k$, then $x$ does not belong to $S$ and is adjacent to a vertex $z \in D_i$. If $z = v$, then there exists a vertex $y$ of $S$ such that $d(u, y) = i$; we have $y \in D_i$ and $d(x, y) = i \leq k$. If $z \neq v$, $i \neq 0$, $i \neq 1$, $i \neq 2$, then there exists a vertex $y$ of $S$ such that $d(u, y) = d(u, z) - k + i - 2$; we have $d(x, y) = k - i + 3 \leq k$. If $i = 1$, then we have $z \in D_i$ and $d(x, z) = 1 \leq k$. If $i = 0$ or $i = 2$, then the vertex $y$ of $S$ adjacent to $z$ has the property that $y \in D_i$ and $d(x, y) = 2 \leq k$.

Now suppose $i > k$. Let again $x \in V(T_m) - D_i$, then $x \in D_j$ for some $j \neq i$. If $j \leq k$, then there exists a vertex $z$ of $S$ such that $d(u, z) = d(u, x) - j + 1$; we have $z \in D_i$ and $d(x, z) = j - 1$. There exists a vertex $y \in D_i$ adjacent to $z$ and $d(x, y) = j \leq k$. If $j > k$, then $x$ is adjacent to a vertex $z \in D_i$ and there exists another vertex $y$ adjacent to $z$ such that $y \in D_i$; while $d(x, y) = 2 \leq k$.

Thus we have proved that each $D_i$ is a $k$-dominating set in $T_m$ and $\{D_0, D_1, \ldots, D_{m-1}\}$ is a $k$-domatic partition of $T_m$, which implies $d_k(T_m) \geq m$. Now let $w$ be the terminal vertex of $S$ distinct from $u$. There are exactly $m$ vertices (including $w$ itself) whose distance from $w$ in $T_m$ is less than or equal to $m$. By the same consideration as in the proof of Theorem 2 we prove that $d_k(T_m)$ cannot be greater than $m$ and thus $d_k(T_m) = m$. \[\square\]

In Fig. 1 there is a tree $T_m$ for $k = 4, m = 7, n = 23$.

![Fig. 1.](attachment:image.png)

**Theorem 4.** Let $C_n$ be a circuit with $n$ vertices, let $k$ be a positive integer. Then

$$d_k(C_n) = \left\lceil \frac{n}{2k + 1} \right\rceil.$$ 

**Proof.** If $n < 2k + 1$, then $d_k(C_n) = n$ according to Proposition 2 and

$$n = \left\lceil \frac{n}{2k + 1} \right\rceil.$$ 

If $n \geq 2k + 1$, then to each vertex of $C_n$ there exist exactly $2k + 1$ vertices (including this vertex itself) whose distances from this vertex are at most $k$. Therefore each $k$-dominating set in $C_n$ has at least $\lceil n/(2k + 1) \rceil$ vertices and each domatic partition
of $C_n$ has at most

$$\left\lfloor \frac{n}{2k + 1} \right\rfloor$$
classes.

Now denote

$$q = \left\lfloor \frac{n}{2k + 1} \right\rfloor, \quad r = (2k + 1)q - n, \quad s = \left\lceil r/q \right\rceil.$$

The circuit $C_n$ can be divided into $q$ edge-disjoint paths such that $qs - r$ of them have the length $2k + 2 - s$ and the remaining $q + r - qs$ of them have the length $2k + 1 - s$. (The reader may verify that $qs - r < q$ and that the sum of the lengths of the described paths is equal to $n$.) Let $P$ be the set of the described paths. We colour the vertices of $C_n$ by the colours $0, 1, \ldots, 2k - s$ in the following way. The terminal vertices of the paths of $P$ (each of them common for two of these paths) are coloured by 0. Now we choose a sense of running around $C_n$. If a path from $P$ has the length $2k + 1 - s$ (or $2k + 2 - s$), we run along it in the chosen sense and colour its inner vertices consecutively by the colours $1, \ldots, 2k - s$ (or $0, 1, \ldots, 2k - s$, respectively). Let $D_i$ be the set of vertices of $C_n$ which are coloured by the colour $i$ for $i = 0, 1, \ldots, 2k - s$. We see that for any fixed $i$ the distance between two vertices of $D_i$ is at most $2k + 2 - s$ for $s \geq 1$ and at most $2k + 1 - s$ for $s = 0$; thus in both the cases at most $2k + 1$. This implies that any vertex not belonging to $D_i$ has the distance at most $k$ from some vertex of $D_i$. Hence $D_i$ is a $k$-dominating set in $C_n$, $\{D_0, D_1, \ldots, D_{2k-s}\}$ is a $k$-domatic partition of $C_n$ and $d_k(C_n) \geq 2k - s + 1$. We shall compute $2k - s + 1$. We have

$$2k - s + 1 = 2k - \left\lceil r/q \right\rceil + 1 = 2k - \left\lceil (2k + 1)q - n)/q \right\rceil + 1 =$$

$$= 2k - (2k + 1) + [n/q] + 1 = [n/q] = \left\lfloor \frac{n}{2k + 1} \right\rfloor.$$

Therefore $d_k(C_n)$ is greater than or equal to this number; as the converse inequality was proved above, it is equal to it. □

References


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