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THE SEMIGROUP OF FINITE COMPLEXES OF A GROUP

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INTRODUCTION

Let G be a group, and denote by $F(G)$ the collection of finite complexes (subsets) of G. $F^*(G)$ denotes $F(G) \setminus \phi$. $F(G)$ is a semigroup under the operation

$$AB = \{ab \mid a \in A,\ b \in B\}$$

for all $A, B \in F(G)$, and also $F(G)$ is a lattice under the operations of set union and set intersection. In fact $F(G)$ is a lattice ordered semigroup ([11], p. 153), since the multiplication preserves order and $A(B \cup C) = AB \cup AC$, $(B \cup C) A = BA \cup CA$. It is easily seen that the dual law $A(B \cap C) = AB \cap AC$ is not satisfied by $F(G)$. Such semigroups have been used as examples for a long time (cf., [11] p. 156) and recenlty have been the subject of further study in several contexts. As one example the concept of retraction was introduced and studied by Byrd, Lloyd, Mena, Teller in [2], [3], [4]. A retraction is a semigroup homomorphism $\sigma: F^*(G) \to G$ such that $\sigma(\{g\}) = g$ for all $g \in G$. The automorphism group of $F^*(G)$ has been examined in [5], [6], [7], [8]. V. Trnková in [12] has considered the problem of embedding a semigroup into 2^G, for some group G. It is because of this recent interest in $F(G)$ that we have in this paper studied $F(G)$ algebraically in an attempt to gain a better understanding of this semigroup and possibily provide some additional tools with which to work.

The first section is devoted to general information concerning factorization and irreducible elements. The second section introduces the concept of an AL-semigroup and uses this definition to give one characterization of $F(G)$. Some examples of AL-semigroups are also given.

I. GENERAL PROPERTIES

It is noted first that $F(G)$ is not cancellative as a semigroup. For example, consider the following sets in $F(Z)$:

$$\{0, 1, 2\} + \{0, 1\} = \{0, 2\} + \{0, 1\}\,.$$

We shall say $A \in F(G)$ is *irreducible* if $|A| \geqq 2$ and $A = BC$ implies that B or C is a unit.

Lemma 1. *If G is a torsion free group, $A \in F(G)$, $|A| \geqq 2$, $A = BC$, B and C not units, then $|B|$, $|C| \leqq |A| - 1$.*

Proof. By way of contradiction, assume $|B| = |A|$ and $|C| \geqq 2$. Let $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$, $C = \{c_1, c_2, \dots, c_k\}$. Since $b_1c_1, b_2c_1, \dots, b_nc_1$ are all distinct elements in A we may assume $a_1 = b_1c_1$, $a_2 = b_2c_1, \dots, a_n = b_nc_1$. The same reasoning gives $b_1c_2, b_2c_2, \dots, b_nc_2$ equal to the a_i in a possibly different order. Setting the two representations equal accordingly we get $b_{ji}^{-1}b_i = c_2c_1^{-1} = c$ for $j = 1, \dots, n$. The b_{ji} are the rearranged b_i, $i = 1, \dots, n$. Form a product with the first element equal to $b_{j1}^{-1}b_1$ and the second element equal to $b_{jk}^{-1}b_k$ where $b_{jk}^{-1} = b_1^{-1}$. Since each b_i appears exactly once we can continue this choice for $i = 1, \dots, n$. The result will be c^n and, also, $c^n = e$. Thus, c is of finite order and we contradict our assumption that C had at least two elements.

Theorem 2. *If G is a torsion free group and $A \in F(G)$ such that $|A| \geqq 2$, then $A = P_1P_2 \dots P_k$ where each P_i is irreducible.*

Proof. Use induction on the cardinality of A. If A is not irreducible, then $A = BC$ where B and C are not units. From the Lemma we know $|B|$, $|C|$ are less than $|A|$, thus, $B = P_1 \dots P_t$, $C = Q_1 \dots Q_r$ where each P_i, Q_j are irreducible. Therefore A can be written as a product of irreducibles.

Corollary 3. *Let H be a subgroup of a torsion free group G. If $A \in F(H)$, $|A| \geqq 2$, $A = P_1 \dots P_k$, $P_i \in F(G)$ and each P_i is irreducible in $F(G)$, then $A = P'_1 \dots P'_k$, $P'_i \in F(H)$ and each P'_i is irreducible in $F(G)$.*

Proof. $A = BC = \{b_1, \dots, b_n\}\{c_1, \dots, c_m\}$, each $b_ic_j \in H$. Then $A = B'C' = \{b_1c_1, \dots, b_nc_1\}\{c_1^{-1}c_1, \dots, c_1^{-1}c_n\}$ and $B', C' \in F(H)$. This method will work for any finite number of irreducible elements $P_1, \dots, P_n$ and, moreover, since the resulting P'_i are translates of the P_i they are irreducible.

For a torsion free group G, $F(G)$ does not have unique factorization into irreducibles. Consider the following sets in $F(Z)$.

$$\{0, 1\} + \{0, 1\} + \{0, 1, 4\} = \{0, 2, 3\} + \{0, 1, 3\}.$$

In the case where G is finite and abelian, then A^n is a subgroup of G for some $n \geqq 1$ and all $A \neq \emptyset$. In general, finite subgroups will be idempotents in $F(G)$, whereas in the torsion free case, the only idempotents are $\{e\}$ and $\emptyset$.

It is also of interest to note hat $F(G)$ can be considered as a normed lattice using as a norm the cardinality of each set $A \in F(G)$.

II. ALGEBRAIC CHARACTERIZATION OF $F(G)$

Definition 1. A *partially ordered semigroup* (p.o. semigroup) is a set S with an associative binary operation and a partial order relation $\leqq$ such that

$$a < b\,, \quad a, b, c \in S \Rightarrow ac \leqq bc\,, \quad ca \leqq cb \quad ([11], \text{p. } 153)\,.$$

Definition 2. An *AL-Semigroup* is a p.o. semigroup S with a non-empty subset A satisfying:

(i) $(S, \leqq)$ is an atomistic lattice with a set of atoms A.
(ii) A is a subgroup of S.
(iii) Let e denote the identity for A. If $e \leqq xy$ for $x, y \in S$, then $\exists b \in A$ such that $b \leqq x$, $b^{-1} \leqq y$.
(iv) $A_x = \{a \in A \mid a \leqq x\}$ is finite for each $x \in S$.

A lattice is atomistic if each element $x \neq 0$ is the join of the atoms under it. Condition (iii) of the definition for AL-semigroups may be thought of as a Riesz interpolation property. It is well known that an atomistic, distributive lattice can be embedded in 2^A where A is the set of atoms for the lattice. The authors have attempted to incorporate these ideas into the definition for an AL-semigroup in order to characterize structures similar to $F(A)$.

Lemma 4. *In an AL-semigroup S the condition* (iii) *is equivalent to*: $a \leqq xy$, $a \in A$, $x, y \in S \Rightarrow \exists a_1, a_2 \in A$ *such that* $a = a_1a_2$, $a_1 \leqq x$, $a_2 \leqq y$.

Lemma 5. *In an AL-semigroup S the following properties hold*:

1. *If* $A_x = A_y$, *then* $x = y$.
2. *For all* $x, y \in S$, $A_x \cap A_y = A_{x \wedge y}$.
3. *If* $b \in A$, $x \in S$, *then* $bx = b(\vee A_x) = \vee\{ba \mid a \in A_x\}$.
4. *If* $a \in A$, $x, y \in S$, *then* $a(x \wedge y) = ax \wedge ay$, $(x \wedge y)\,a = xa \wedge ya$.
5. *If* $a \in A$, $x, y \in S$, *then* $a(x \vee y) = ax \vee ay$, $(x \vee y)\,a = xa \vee ya$.
6. $A_{ax \wedge ay} = \{ab \mid b \in A_x \cap A_y = A_{x \wedge y}\}$ *for all* $a \in A$, $x, y \in S$.
7. *If* $x \wedge y = 0$, $x, y \in S$, *then* $ax \wedge ay = 0$ *for all* $a \in A$.

The proofs of Lemmas 4 and 5 are straightforward.

Theorem 6. *If S is an AL-semigroup, then S is o-isomorphic to a subsemigroup of $F(A)$. The mapping may not preserve joins.*

Proof. Define $\theta: S \to F(A)$ by $\theta(x) = A_x$ for $x \neq 0$ and $\phi(0) = \emptyset$. It follows, from Lemma 4, that $A_{xy} = A_xA_y$. Thus, θ is a semigroup homomorphism. From Lemma 5, (1), it follows that θ is one-to-one. If $x \leqq y$, then $A_x \leqq A_y$. So θ is an order preserving semigroup isomorphism. Examples 1, 2 given later in this section both illustrate embeddings for which joins are not preserved.

It is to be noted that if S is an AL-semigroup embedded in $F(A)$ such that joins in S agree with joins in $F(A)$, then $S = F(A)$. This follows because S is atomistic. The following theorem gives some equivalent conditions for S to equal $F(A)$.

In what follows we shall frequently identify an AL-semigroup with its image in $F(A)$. By (2) of Lemma 5, $C \wedge B = C \cap B$ under this identification.

Theorem 7. *Let S be an AL-semigroup. Then the following are equivalent*:

(1) $(S, \leqq)$ *is a distributive lattice.*
(2) *If* $B \subseteq A$, $\vee B = x \in S$, *then* $B = A_x$.
(3) S *is o-isomorphic as a p.o. semigroup and as a lattice to* $F(A)$.
(4) $A_{x \vee y} = A_x \cup A_y$ *for all* $x, y \in S$.
(5) $(S, \leqq)$ *is a normed lattice with norm* $\|x\| = |A_x|$.

Proof. (1) ⇒ (2): Suppose $\vee B = x$, $B \subset A_x$ and $B \neq A_x$. Let $y \in A_x \backslash B$. $B = \{y_1, \ldots, y_k\}$, $A_x = \{y_1, \ldots, y_k, y, \ldots\}$, $y = y \wedge x = y \wedge (y_1 \vee \ldots \vee y_k \vee y \vee \ldots) = y \wedge (y_1 \vee \ldots \vee y_k) = (y \wedge y_1) \vee \ldots \vee (y \wedge y_k) = 0$. This contradicts the fact that $y \in A_x$. Therefore $A_x = B$.

(2) ⇒ (3): From Theorem 6 we have that $\phi(x) = A_x$ is an o-isomorphism into $F(A)$. Let $B \in F(A)$ and $x = \vee B$. Thus, $B = A_x$ and $\phi(x) = B$. Therefore ϕ is an o-isomorphism onto $F(A)$. To show ϕ is a lattice isomorphism we need $A_{x \vee y} = A_x \cup A_y$ and $A_{x \wedge y} = A_x \cap A_y$. The later equality is true in any AL-semigroup. $A_x \cup A_y \subseteq A_{x \vee y}$ is obvious. Suppose there exists $a \leqq x \vee y$, $a \nleqq x$, $a \nleqq y$. Let $B = A_{x \vee y} \backslash \{a\}$. $x \vee y \geqq \vee B \geqq \vee(A_x \cup A_y) = (\vee A_x) \vee (\vee A_y) = x \vee y$. Therefore by (2), we have $B = A_{x \vee y}$, but this is a contradiction.

(3) ⇒ (4): This is valid for any $F(A)$.

(4) ⇒ (5): We need to verify the equality for a normed lattice: $\|x \vee y\| + \|x \wedge y\| = \|x\| + \|y\|$. This translates into $|A_{x \vee y}| + |A_{x \wedge y}| = |A_x| + |A_y|$. Since $A_{x \vee y} = A_x \cup A_y$ and $A_{x \wedge y} = A_x \cap A_y$, the desired equality holds.

(5) ⇒ (1): We have $|A_{x \vee y}| + |A_{x \wedge y}| \geqq |A_x \cup A_y| + |A_x \cap A_y| = |A_x| + |A_y|$. Since $|A_x|$ is a norm on S, $A_{x \vee y} = A_x \cup A_y$. Since a normed lattice is modular, we may therefore assume there exists $x, y, z \in S$ such that $x \vee y = x \vee z = y \vee z$ and $x \wedge y = x \wedge z = y \wedge z$. Since $A_{x \vee y} = A_{x \vee z} = A_x \cup A_y = A_x \cup A_z$ we have that $A_z \subseteq A_x \cup A_y$. Assume $a \in A_x$, $a \notin A_z$. Then since $A_x \subseteq A_y \cup A_z$, $a \in A_y$; $a \in A_x \wedge A_y = A_{x \wedge y} = A_{x \wedge z}$. Therefore $a \leqq z$ which is a contradiction. Since there is no such a we have that $A_z = A_x \cup A_y$. Therefore $\vee A_z = z = \vee(A_x \cup A_y) = \vee A_{x \vee y} = x \vee y$. This contradicts our choice of x, y, z. Thus, S is distributive.

Example 1. Let G be a finite group with $|G| \geqq 3$. Let S equal all singleton subsets of G together with G and the empty set. This is a modular, non-distributive AL-semigroup.

Example 2. Let Z be the group of integers. Any set $A \in F(Z)$ can be written as $A = \{x_1, \ldots, x_n\}$ such that $x_1 < x_2 < \ldots < x_n$. Let S be the sets A such that if $z \in Z$

and $x_1, x_n \in A$ such that $x_1 \leqq z \leqq x_n$, then $z \in A$. These are called *solid sets* in $F(Z)$. S is a non-modular AL-semigroup. To illustrate the non-modularity consider the sets $\{0, 1, 2\}, \{1, 2\}, \{2\}, \{0\}, \emptyset$.

Example 3. Let H be a subgroup of G. Then $F(H)$ is an AL-semigroup contained in $F(G)$.

Theorem 7 characterizes distributive AL-semigroups. Examples 1 and 2 give instances of non-distributive AL-semigroups, the first of which is modular and the second of which is not. It seems natural then to try to characterize modular AL-semigroups. A partial description is given in Theorem 11, where we prove that if S is an AL-semigroup with torsion free abelian group of atoms, then S is modular only if S is distributive. To prove Theorem 11 we shall use the following.

Lemma 9. *Let S be an AL-semigroup which is modular. If S is not distributive, then exist distinct atoms a, b, c such that $a \vee b = a \vee c = b \vee c$.*

Proof. Since S is not distributive, there exists $x, y \in S$ such that $A_{x \vee y} \supset A_x \cup A_y$ and $A_{x \vee y} \neq A_x \cup A_y$. Choose $x, y \in S$ such that $|A_{x \vee y}|$ is minimal with respect to the property $A_{x \vee y} \supset A_x \cup A_y$ and $A_{x \vee y} \neq A_x \cup A_y$. Let $a \in A_{x \vee y} \backslash A_x \cup A_y$. Then $a \vee y \leqq x \vee y$, and so $|A_{a \vee y}| \leqq |A_{x \vee y}|$. If $a \vee y < x \vee y$, then $A_{a \vee y} = A_y \vee \{a\}$. Also $A_{(y \vee a) \wedge x} = (A_y \cup \{a\}) \cap A_x = A_y \cap A_x = A_{y \wedge x}$. Therefore, $(y \vee a) \wedge x = y \wedge x$. So $y \vee x$, x, $y \vee a$, y, $y \wedge x$ will form a non-modular sublattice. Thus $a \vee y = x \vee y$. Dually $a \vee x = x \vee y$. Since $a \notin A_x \cup A_y$, $a \wedge (x \wedge y) = 0$.

Since $x > x \wedge y$, $\exists b \in A_x \backslash A_{x \wedge y}$. Suppose now that $(b \vee a) \wedge y = 0$. Then $(b \vee a) < b \vee a \vee y$ since otherwise $b \vee a = b \vee a \vee y$ implies $(b \vee a) \wedge y = y \neq 0$. Therefore we have a non-modular lattice formed by $b \vee a \vee y$, $b \vee a$, y, a, 0. Therefore, $(b \vee a) \wedge y > 0$. Choose $c \in A$ such that $c \in A_y$, $c < b \vee a$. Now assume $a \vee c < a \vee b$. Note that $(a \vee c) \wedge b = 0$, since otherwise $(a \vee c) \wedge b = b$ and $b \leqq a \vee c$. Thus, $a \vee c \vee b = a \vee c \geqq a \vee b$ and this contradicts our assumption that $a \vee c < a \vee b$. We also have that $(a \vee c) \wedge b = 0$. It follows that $a \vee b$, $a \vee c$, a, b, 0 will form a non-modular lattice. By a dual argument we can get $b \vee c = a \vee b = a \vee c$. Now we consider the atoms $a \in A_{x \vee y} \backslash A_x \cup A_y$, $b \in A_x \backslash A_{x \wedge y}$, $c \in A_y$ such that $c < b \vee a$. These atoms form the desired sublattice of S.

In the next lemma we make use of the notion of height in a modular lattice.

Definition. In a modular lattice L the *height of an element* a is the sup of the integers n for which there is a chain $a_0 < a_1 < \ldots < a_n = a$ ([1], p. 5). Since L is modular, this is the length of any maximal chain in $[0, a]$; if L is atomistic this is the minimum number of atoms which must be joined to get a.

Lemma 10. *Let S be a modular AL-semigroup with an abelian group of atoms A. Suppose $x_0 = e$, and $x_0 \vee x_1 = \{x_0, x_1, \ldots, x_n\} = X \subseteq A$, where $n \geqq 2$. If $Xx_i \cap Xx_j = \{x_i x_j\}$ for all i, j, $i \neq j$, then $x_0 \vee x_1^2 = e \vee x_1^2 = \{x_0, x_1^2, \ldots, x_n^2\}$.*

Proof. Since $Xe \cap Xx_1 = \{x_1\}$, $x_1^2 \notin X$. Also X has height 2, so is the join of any two distinct x_i's.

Suppose now that $i \neq k$. Then $Xx_k \vee x_jx_i \supseteq x_jx_k \vee x_jx_i = x_j(x_k \vee x_i) = x_jX$. If $i, j \neq k$, and $m \in \{0, 1, \ldots, n\}$, $x_mx_i \in x_iX \subseteq x_ix_j \vee Xx_k$. Then $x_ix_j \vee Xx_k \supseteq \supseteq x_iX \vee Xx_k \supseteq x_mx_i \vee Xx_k \supseteq Xx_m$.

Observe that $\bigcup_{i=0}^{n} Xx_i = X^2$. Then $X^2 \subseteq x_ix_j \vee Xx_k \subseteq X^2$, so $X^2 = x_ix_j \vee Xx_k = x_ix_j \vee x_k \vee x_k^2$. Thus X^2 has height at most 3; but since $X \subset X^2$, $X \neq X^2$ and X has height 2, we see that X^2 has height 3. It follows that for any atoms x_ix_j, x_kx_m, x_px_q, if $x_ix_j \nleq x_kx_m \vee x_px_q$ then $X^2 = x_ix_j \vee x_kx_m \vee x_px_q$. Also, if Y, Z are any sets of height 2, $Y, Z \leqq X^2$, then $Y \wedge Z$ must be an atom ($Y \wedge Z = \emptyset$ gives a non-modular lattice consisting of $Y, Z, X^2, \emptyset$ and any atom $y \leqq Y$).

We now fix a set Y of height 2, $Y < X^2$. Write $Y = \{y_1, \ldots, y_m\}$. We next wish to show that $m \geqq n + 1$, so suppose that in fact $m \leqq n$. $Y \cap X$ is an atom, so let x_k be some element of X not in $Y \cap X$. Now pick x_j such that Xx_j is not $y_r \vee x_k$ for any $r = 1, \ldots, m$. This is possible because $Xx_i \neq Xx_j$ for $i \neq j$ and there are only m sets $y_r \vee x_k$, but $n + 1$ sets Xx_i. Now define a map $Y \to Xx_j$ by $y_r \to (y_r \vee x_k) \wedge \wedge Xx_j$. The image of each y_r is an atom. The image of Y has at most m elements and is contained in Xx_j which has $n + 1$ elements, so there exists an $x_jx_s \in Xx_j$, $x_jx_s \notin \notin y_r \vee x_k$ for any r. Let $Y_1 = x_jx_s \vee x_k$. Clearly $x_jx_s \neq x_k$, since $x_k \in y_r \vee x_k$ for all r. Thus Y_1 has height 2. Therefore $Y \cap Y_1$ is an atom $y_i \in Y$. But $y_i \in Y_1$, $y_i \neq x_k$, so $y_i \vee x_k = Y_1$ and $x_jx_s \in y_i \vee x_k$, a contradiction. Thus we must have $|Y| \geqq n + 1$.

Clearly $e \vee x_1^2 \leqq X^2$. Note that $e \vee x_1^2 \neq Xx_i$ for any i, for otherwise $x_1^2 \in Xx_i \cap \cap Xx_1 = \{x_1x_i\}$, so $i = 1$, and thus $e \in X \cap Xx_1 = \{x_1\}$, a contradiction. Then $e \vee x_1^2$ has height 2, and $(e \vee x_1^2) \cap Xx_i$ will be an atom for each $i = 0, 1, \ldots, n$. Further, $e \vee x_1^i = \bigcup_i [(e \vee x_1^2) \cap Xx_i]$, and each set in the union is an atom. If $e \vee x_1^2$ contains an element x_ix_j, $i \neq j$, $(e \vee x_1^2) \cap Xx_i = (e \vee x_1^2) \cap Xx_j$, and $e \vee \vee x_1^2$ will contain fewer than $n + 1$ elements. Thus $e \vee x_1^2 \cap Xx_i = x_i^2$, and $e \vee x_1^2 = = \{x_0^2, x_1^2, \ldots, x_n^2\}$.

Theorem 11. *Is S is a modular AL-semigroup with the group of atoms abelian and torsion free, then S is distributive.*

Proof. Since any abelian torsion free group can be totally ordered ([11], p. 36), we assume A is totally ordered. By Lemma 9, there exist atoms a, b, c such that $a \vee b = a \vee c = b \vee c$. Let $X = \{x_1, x_2, \ldots, x_n\} = a \vee b$. Assume $x_1 < x_2 < \ldots \ldots < x_n$. $x_1^{-1}X = \{e, x_1^{-1}x_2, \ldots, x_1^{-1}x_n\} = \{e, y_2, \ldots, y_n\}$ where $e < y_2 < \ldots < y_n$. Moreover, $y_i \vee y_j = x_1^{-1}X$ for $i \neq j$. X has height 2, thus, $x_1^{-1}X$ has height 2. Therefore we can assume $a \vee b = \{e, x_1, \ldots, x_n\}$, $n \geqq 2$, $e = x_0 < x_1 < \ldots < x_n$ and $x_i \vee x_j = X$ for $i \neq j$. We have $x_ix_j \in Xx_i \cap Xx_j$. Suppose there exists $x_ix_k = = x_jx_l \in Xx_i \cap Xx_j$, $l \neq i$, $k \neq j$. Then $x_ix_j \vee x_ix_j = x_iX = x_ix_k \vee x_ix_j = x_ix_l \vee \vee x_jx_i = x_jX$. Since X is ordered with minimum equal to e, $\min(Xx_j) = x_j = x_i = = \min(Xx_i)$. Therefore, $Xx_i \cap Xx_j \cap \{x_ix_j\}$ for $i \neq j$. We can now apply Lemma

10. We have $e \vee x_1^2 = Y = \{e, x_1^2, \dots x_n^2\}$, $X \neq Y$, $X \vee Y = X^2$ and $XY = \{x_i x_j^2 \mid i = 0, 1, \dots, n;\ j = 0, 1, \dots, n\} \supseteq X$ and Y. $X \vee Y = X^2 \subseteq XY$. Therefore $x_1 x_2 \in XY$, $x_1 x_2 = x_i x_j^2$. A case study for $i = 0$, $j = 0, 1, \dots, n$ will show that $i = 0$ is not possible. Also the case where $i = 1$ and $j = 0, 1, \dots, n$ is not possible. Then consider the case where $j = 0$ and $i = 2, \dots, n$ which is also impossible. Using the fact that $x_0 < x_1 \dots < x_n$ it then follows that $x_1 x_2 < x_i x_j^2$, contrary to our choice of $x_i x_j^2$.

As was indicated in the discussion prior ot Lemma 9, if we assume that A is a torsion free abelian group, then Theorem 11 allows Theorem 7 to be rephrased so as to give a characterization for modular AL-semigroups.

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