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TOLERANCES ON MEDIAN ALGEBRAS

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Several articles were devoted to the study of "tolerances", i.e. reflexive and symmetric compatible relations on algebras. For distributive lattices, tree algebras,and more generally, median algebras, one understands these relations well. The purpose of this note is to give the main facts, provide simple proofs, and to extend some previous results concerning tolerances.

A meet-semilattice M is called a *median semilattice* if every principal ideal is a distributive lattice and any three elements of M have an upper bound whenever each pair of them does. For any three elements a, b, c in a median semilattice M we can define their *median* (abc) by

$$(abc) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) .$$

This ternary operation has the following properties:

$$(aab) = a ,$$
$$(abc) = (bac) = (bca) ,$$
$$((abc)\, de) = (a(bde)\,(cde)) .$$

Conversely, a ternary algebra satisfying these identities is called a *median algebra*. Every median algebra M admits semilattice orders $\leqq$ (so-called *compatible* semilattice orders) for which $(M, \leqq)$ is a median semilattice, the median being the given ternary operation (see [8], [9]). In particular, for any element a of M one can define a compatible semilattice order $\leqq_a$ by $x \leqq_a y \Leftrightarrow x = (axy)$ (see [8]). For a survey of median algebras and a full description of their compatible semilattice orders, see [1].

Let M be a median algebra. For any $a, b \in M$, the set $a \circ b = \{x \in M \mid x = (abx)\}$ is called a *segment* (or an interval). A subset A of M is *convex* in M (or is an ideal) if $(abc) \in A$ for all $a, b \in A$ and $c \in M$. A convex set $A \neq \emptyset$, M is called *prime* if, in addition, the complement $M - A$ of A is convex. By [7] Theorem 1 for every convex set A in M with $x \notin A$, there exists a prime convex set P such that $A \subseteq P$ and $x \notin P$.

By a *tolerance* on a median algebra M we mean a reflexive and symmetric subalgebra of $M^2 = M \times M$. A subset B of M is called a *block* of a reflexive and symmetric relation ξ on M if B is maximal with respect to the property $B^2 \subseteq \xi$. Clearly, every block of a tolerance on M is convex in M (cf. [6] Lemma 2). If A and B are convex sets with $A \cup B = M$, then $A^2 \cup B^2$ is a tolerance on M having A and B as its blocks. Given a non-empty convex set C in M, let $\theta[C]$ denote the least congruence on M with block C (cf. [1]).

Every distributive lattice is a median semilattice and thus gives rise to a median algebra. For a distributive lattice, median notions correspond to lattice notions in the obvious way (cf. [1] Proposition 1.3). In particular, a tolerance on a distributive lattice L (i.e. a reflexive and symmetric sublattice of L^2) is a tolerance on the corresponding median algebra, and vice versa. Hence Propositions 1, 3, 4 below immediately apply to tolerances on distributive lattices.

Proposition 1. *Let M be a median algebra and let ξ be a tolerance on M. Then ξ is the union of the congruences $\theta[C]$, where C runs through all blocks of ξ. Further, ξ is the intersection of tolerances of the form $P^2 \cup Q^2$, where P, Q are prime convex sets in M with $P \cup Q = M$.*

Proof. By [1] Theorem 1.6, $(x, y) \in \theta[C]$ if and only if there exist $a, b \in C$ such that $x = (axy)$ and $y = (bxy)$. Hence if C is a block of ξ, then $\theta[C] \subseteq \xi$. To prove the second assertion, first observe that for every convex set A in M, $\xi(A) = \{x \in M \mid (a, x) \in \xi \text{ for some } a \in A\}$ is a convex set in M. So, for every $(x, y) \notin \xi$ we can find a prime convex set P in M with $\xi(\{y\}) \subseteq P$ and $x \notin P$. Since $y \notin \xi(M - P)$, there is a prime convex set Q in M such that $\xi(M - P) \subseteq Q$ and $y \notin Q$. Hence, $\xi \subseteq P^2 \cup Q^2$ and $(x, y) \notin P^2 \cup Q^2$, as desired.

Note that Proposition 1 implies [4] Theorem 1, Corollary 3, and [5] Proposition 1.

The block structure of tolerances on median algebras was discussed in [6]. However, [6] Lemma 3 is incorrect: consider the five element lattice $L = \{0, a, b, c, 1\}$ with $a \vee b = c$; the blocks of the congruence collapsing c and 1 do not form a subalgebra of the median algebra of all convex sets.

A *tree algebra* is a median algebra M in which any (abd), (acd), (bcd) are not distinct (cf. [8]). A *tree semilattice* is a meet-semilattice in which any two incomparable elements do not have an upper bound. Clearly, a tree semilattice is a median semilattice, and a tree algebra is a tree semilattice with respect to any compatible semilattice order (cf. [1], [9]). If M is a tree algebra, then $a \circ b \subseteq (a \circ x) \cup (b \circ x)$ for all $a, b, x \in M$; hence the union of convex sets containing x is again a convex set. Tolerances on tree algebras are readily characterized (for the finite case, see [10]):

Proposition 2. *Let ξ be a reflexive and symmetric relation on a tree algebra M. Then ξ is a tolerance if and only if all blocks of ξ are convex.*

Proof. Let ξ be a reflexive and symmetric relation on M such that every block of ξ is convex. Suppose that $(a, b) \notin \xi$. By [1] Theorem 6.4, $A = \{x \in M \mid b \notin a \circ x\}$ is a prime convex set in M. The union B of the set $M - A$ and all blocks of ξ containing

b is a convex set and does not contain a, whence $(a, b) \notin A^2 \cup B^2$. Every block C of ξ which intersects both A and $M - A$ must contain b; for, if $x, y \in C$ with $x \in M - A$ and $y \in A$, then $(axy) = (aby)$ and so $b = ((aby)\, bx) = ((axy)\, bx) = ((abx)\, xy) = (bxy) \in C$. We infer that $\xi \subseteq A^2 \cup B^2$, and therefore ξ is a tolerance.

A median algebra M is said to have the *Tolerance Extension Property* if for a tolerance ζ on a subalgebra S of M there exists a tolerance ξ on M such that $\xi \cap S^2 = \zeta$ (cf. [2]). Now, let M be a tree algebra and let ζ be a tolerance on a subalgebra S of M. Then by Proposition 2, the union of all sets $(a \circ b)^2$ for which either $(a, b) \in \zeta$ or $a = b$ in M is a tolerance on M (extending ζ). Since the four element Boolean lattice does not have the Tolerance Extension Property, we arrive at

Corollary. *A median algebra M has the Tolerance Extension Property if and only if M is a tree algebra.*

For every median algebra M, the maximal relatively complemented convex sets in M form the blocks of a tolerance on M. To prove this, we use [1] Theorem 6.1 for some straightforward characterizations of relatively complemented distributive lattices in terms of their medians.

Proposition 3. *For a median semilattice $(M, \leqq)$, let β denote the intersection of all $P^2 \cup Q^2$, where P and Q are prime convex sets in M such that $P \cup Q = M$ and $P \cap Q \neq \emptyset$. Then, $(x, y) \in \beta$ if and only if the segment $x \circ y$ is a Boolean lattice (with respect to $\leqq$). Consequently, the blocks of β are the maximal convex sets which are relatively complemented distributive lattices (with respect to $\leqq$).*

Proof. Suppose that $x \circ y$ is a Boolean segment in M. Let P, Q be prime convex sets in M such that $P \cup Q = M$ and $P \cap Q \neq \emptyset$, say $a \in P \cap Q$. Assume that $x \in P - Q$ and $y \in Q - P$. Then $u = (axy) \in P \cap Q$, and since $x \circ y$ is Boolean, there exists $v \in x \circ y$ such that $x, y \in u \circ v$. Since either $u \circ v \subseteq P$ or $u \circ v \subseteq Q$, we arrive at a contradiction. Therefore $(x, y) \in P^2 \cup Q^2$. Conversely, if $x \circ y$ is not Boolean, then there exist convex sets A, B in $x \circ y$ such that $A \cup B = x \circ y$, $A \cap B \neq \emptyset$, $x \in A - B$, $y \in B - A$. Now, there exists a prime convex set P in M with $A \subseteq P$ and $y \notin P$. The convex set C generated by B and $M - P$ does not contain x; for otherwise, by [7] Lemma 2 we must have $x = (bxz)$ for some $b \in B$ and $z \in M - P$, whence $x = ((bxz)\, xy) = (bx(xyz)) \in B$ (because $(xyz) \notin A$), which is absurd. So, there exists a prime convex set Q in M with $C \subseteq Q$ and $x \notin Q$. Consequently, $P \cup Q = M$, $\emptyset \neq A \cap B \subseteq P \cap Q$, and $(x, y) \notin P^2 \cup Q^2$.

The following proposition generalizes [3] Theorem 16 and [6] Theorem 1.

Proposition 4. *For every median algebra M, the lattice $\Xi(M)$ of all tolerances on M is distributive.*

Proof. Let $\xi_0, \xi_1, \xi_2 \in \Xi(M)$ and $(a, b) \in \xi_0 \cap (\xi_1 \vee \xi_2)$. Since $\xi_1 \vee \xi_2$ is the subalgebra of M^2 generated by $\xi_1 \cup \xi_2$, we can find an n-ary polynomial p and pairs

$(a_i, b_i) \in \xi_1 \cup \xi_2$ $(i = 1, \ldots, n)$ such that $a = p(a_1, \ldots, a_n)$ and $b = p(b_1, \ldots, b_n)$. Induction on n shows that the function $x \to (abx)$ is distributive over p, whence $a = (aba) = p((aba_1), \ldots, (aba_n))$ and $b = (abb) = p((abb_1), \ldots, (abb_n))$. For $i = 1, \ldots, n$, $((aba_i), (abb_i)) \in (a \circ b)^2 \subseteq \xi_0$ and $((aba_i), (abb_i)) \in \xi_1 \cup \xi_2$, whence $((aba_i), (abb_i)) \in (\xi_0 \cap \xi_1) \cup (\xi_0 \cap \xi_2)$. We conclude that $(a, b) \in (\xi_0 \cap \xi_1) \vee (\xi_0 \cap \xi_2)$, proving that $\Xi(M)$ is distributive.

In closing, we mention that the tolerance lattice of the free median algebra on three generators is the free distributive lattice on three generators.

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