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APPLICATION OF MOSER'S METHOD TO A CERTAIN TYPE OF EVOLUTION EQUATIONS

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1. INTRODUCTION

Moser's method was developed as a tool for investigating problems in which the usual iterative procedures fail. This is due to a phenomenon which may be described as "loss of derivatives". This phenomenon occurs e.g. in the case of operators $L_p u = u_{tt} + (-1)^p u_{x^{2p}} + \alpha u_t$, which map the spaces $H_{k,pk}$ (functions with k generalized derivatives with respect to t and pk with respect to x) into $H_{k-2,p(k-2)}$, but the inverse operators map $H_{k-2,p(k-2)}$ into $H_{k-1,p(k-1)}$. This is an inherent difficulty in treating the nonlinear equation $L_p u = F(u)$, where $F(u)$ includes the derivatives of u up to the same order as L_p.

The problem $L_1 u = \varepsilon f(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$,

$$(1) \qquad u(t + 2\pi, x) = u(t, x), \quad u(t, 0) = u(t, \pi) = 0$$

has been investigated by P. H. Rabinowitz [2]. The problem $L_2 u = \varepsilon F(u)$, (1) where $F(u) = f(t, x, u, u_t, u_x, u_{tt})$ or $F(u) = f(t, x, u, u_t, u_x, u_{x^4})$ is studied in the thesis of M. Štědrý [5].

In this paper a condition is found under which the main assumption of Moser's theorem (see [1]) is fulfilled. Then it is applied to the problem $(L_p - I) u = \varepsilon F(u)$, where, moreover, the bifurcation equation must be solved, because the operators $L_p - I$ have a nontrivial kernel.

2. APPROXIMATE SOLUTIONS OF THE LINEARIZED EQUATION

Let $H_0 \supset \hat{H}_0 \supset H_1 \supset \hat{H}_1 \ldots H_k \supset \hat{H}_k \ldots$ be Hilbert spaces with norms satisfying $|u|_0 \leqq \|u\|_0 \leqq |u|_1 \leqq \ldots \leqq |u|_k \leqq \|u\|_k \ldots$ ($|\cdot|_i$, $\|\cdot\|_i$ denote the norms in H_i, $\hat{H}_i$ respectively), and

$$(2) \qquad |u|_{k+i+1} \leqq c|u|_{k+i}^{1-\tau} |u|_{k+i+2}^{\tau}, \quad 0 < \tau < 1,$$

$$\|u\|_l \leqq c\|u\|_0^{1-\sigma} \|u\|_k^{\sigma}, \quad 0 < l < k, \quad 0 < \sigma < 1.$$

(All constants will be denoted by c). Denote by L a linear operator continuously mapping $H_{l+i} \to H_l$, $\hat{H}_{l+i} \to \hat{H}_l$ for $0 \leqq l \leqq k$ and some $i > 0$ and by F a nonlinear differentiable operator with the same properties.

If we are to prove the existence of a solution in $\hat{H}_l$ to the equation

$$Lu = \varepsilon F(u) \tag{3}$$

with the help of Moser's theorem, it is necessary to verify the following assumptions: Let $u_0 \in \hat{H}_{k+i}$, $D = \{u \in \hat{H}_{k+i}, \|u - u_0\|_l \leqq R\}$, $T(u) = Lu - \varepsilon F(u)$. Then there is a constant $M > 0$ such that

(i) $\|F(u)\|_k \leqq MK$ for $u \in \hat{H}_{k+i} \cap D$, $\|u\|_{k+i} \leqq K$,

(ii) $\|T(u) - T(u_0)\|_0 \leqq M$ for $u \in D$,

(iii) $\|F(u + v) - F(u) - F'(u)\, v\|_0 \leqq M\|v\|_0^{2-\beta}\, \|v\|_{k+i}^{\beta}$ for $u \in D$, $v \in \hat{H}_{k+i}$, $0 < \beta < 1$.

(iv) If $g \in \hat{H}_k$, $u \in \hat{H}_{k+i}$, satisfy $\|g\|_0 < K^{-\lambda}$, $\|g\|_k \leqq K$, $\|u\|_{k+i} \leqq K$, K being sufficiently large, then there exists an approximate solution to the equation $T'(u)\, v = g$, i.e. for every $Q > 1$ there is a $v_Q \in \hat{H}_{k+i}$ such that

$$\|T'(u)\, v_Q - g\|_0 \leqq MKQ^{-\nu}\,,$$
$$\|v_Q\|_{k+i} \leqq MKQ\,,$$
$$\|v_Q\|_0 \leqq c\|T'(u)\, v_Q\|_0\,,$$

(v) $\dfrac{\nu + 1}{\nu - \lambda} < \min\left(2 - \beta\,\dfrac{\lambda + (\lambda + 1)(\nu + 1)}{\lambda(\nu + \beta)}\,,\ \lambda\,\dfrac{1 - \sigma}{\sigma}\right).$

If these assumptions are fulfilled and ε is sufficiently small, (3) has a solution in the space $\hat{H}_l$.

The most difficult task is to verify the assumption (iv). We intend to find conditions for the operators L, F and the spaces, under which (iv) is fulfilled.

Theorem 1. *Let the spaces H_j, $\hat{H}_j$, $0 \leqq j \leqq k + i + 2$, satisfy (2) and let $\{e_n\}$ be an orthogonal basis in each H_j, e_n being the eigenvectors of the operator L. Suppose that there exists a continuous operator $\Lambda : \hat{H}_{l+i} \to H_l$ such that $\Lambda e_n = b_n e_n$, $n = 1, 2, \ldots$ and*

$$(Lv, \Lambda v)_l \geqq c|v|_{l+i}^2\,, \tag{4}$$

$$(F'(u)\, v, \Lambda v)_k \leqq c|v|_{k+i}^2\,, \tag{5}$$

$$(g, \Lambda v)_k \leqq c\|g\|_k\, |v|_{k+i}\,, \tag{6}$$

where $u, v \in \hat{H}_{k+i}$, $\|u\|_{k+i} \leqq K$, $0 \leqq l \leqq k + 2$.

Then the equation $T'(u)\, v = g$ admits an approximate solution.

Proof. We shall prove this theorem in several steps.

A. There is an operator Λ' such that for all $v \in H_{k+i+2}$ and $\mu > 0$

(7) $$|\Lambda' v|_1 \leqq c|v|_{k+i},$$

(8) $$|\Lambda' v|_k \geqq c|v|_{k+i+2},$$

(9) $$(Lv, \Lambda' v)_k \geqq 0, \quad (\Lambda' v, \Lambda v)_k \geqq 0,$$

(10) $(\Lambda + \mu\Lambda')^{-1}$ exists and is continuous from H_k to H_{k+i+2}.

B. The operator $(\mu\Lambda' + T'(u))^{-1} : H_k \to H_{k+i+2}$ exists and the following inequality holds:

(11) $$|v|_{k+i} + \mu|v|_{k+i+2} \leqq c\|(\mu\Lambda' + T'(u))\, v\|_k, \quad \mu > 0.$$

C. The assertion of Theorem 1 follows from relation (11).

Proof of A. As e_n is an orthogonal basis in H_l and $Le_n = a_n e_n$, we get according to (4):

$$(Lv, \Lambda v)_l = \sum_{n=1}^{\infty} a_n \bar{b}_n |v_n|^2 \, |e_n|_l^2 \geqq c \sum_{n=1}^{\infty} |v_n|^2 \, |e_n|_{l+i}^2,$$

where $v_n = (v, e_n)_0$, if we take $|e_n|_0 = 1$. This implies that

(12) $$a_n \bar{b}_n > c|e_n|_{l+i}^2 \, |e_n|_l^{-2} > 0.$$

We define the operator Λ' in the following way: $\Lambda' e_n = c_n e_n$, where c_n are determined to fulfil

(13) $$c_n \bar{b}_n \geqq 0, \quad a_n \bar{c}_n \geqq 0,$$

(14) $$\text{const}\, |e_n|_{k+i} \, |e_n|_1^{-1} \leqq |c_n| \leqq \text{const}\, |e_n|_{k+i} \, |e_n|_1^{-1}.$$

This choice is possible because of (12). It follows from the continuity of the operators L, Λ that

$$a_n \bar{b}_n |e_n|_j^2 = (Le_n, \Lambda e_n)_j \leqq c|e_n|_{j+i+1}^2$$

for each $j \leqq k + 2$. This inequality together with (12) yields

$$|e_n|_{l+i} \, |e_n|_{j+i+1}^{-1} \leqq |e_n|_l \, |e_n|_j^{-1}$$

for arbitrary $l, j \leqq k + 2$ and consequently we obtain

$$|e_n|_{k+i+2} \, |e_n|_{k+i}^{-1} \leqq |e_n|_k \, |e_n|_1^{-1} \quad \text{for} \quad k \geqq \max(6, i + 3).$$

This ensures that for each $g \in H_k$, $g = \sum_{n=1}^{\infty} g_n e_n$ there is a $v \in H_{k+i+2}$,

$$v = \sum_{n=1}^{\infty} g_n c_n^{-1} e_n, \quad |v|_{k+i+2}^2 = \sum_{n=1}^{\infty} |g_n|^2 \, |c_n|^{-2} \, |e_n|_{k+i+2}^2 \leqq$$

$$\leqq c \sum_{n=1}^{\infty} |g_n|^2 \, |e_n|_{k+i+2}^2 \, |e_n|_{k+i}^{-2} \, |e_n|_1^2 \leqq c \sum_{n=1}^{\infty} |g_n|^2 \, |e_n|_k^2 = c|g|_k^2.$$

This and (13_1) imply that (8) and (10) hold, (7) immediately follows from (14) and (9) from (13).

Proof of B. According to (4), (5) and (9) we obtain for an ε sufficiently small

$$(15) \qquad ((\mu\Lambda' + T'(u))\, v, (\mu\Lambda' + \Lambda)\, v)_k \geqq \mu^2 |\Lambda' v|_k^2 + c|v|_{k+i}^2 - \mu\varepsilon |\Lambda' v|_\kappa \, |v|_{k+i} ,$$

$$|(\mu\Lambda' + T'(u))\, v|_k \, (\mu|\Lambda' v|_k + c\|v\|_{k+i}) \geqq c(\mu|\Lambda' v|_k + |v|_{k+i})^2 .$$

With the help of (8) we get

$$|(\mu\Lambda' + T'(u))\, v|_k \, |\Lambda' v|_k \geqq c\, \mu^2 |\Lambda' v|_k^2$$

and finally

$$|(\mu\Lambda' + T'(u))\, v|_k \geqq c\, \mu^2 |\Lambda' v|_k \geqq c\, \mu^2 |v|_{k+i+2} .$$

So the range of the operator $\mu\Lambda' + T'(u)$ is closed in H_k and if $u \in H_k$, $(u, (\mu\Lambda' + T'(u))\, v)_k = 0$ for each $v \in H_{k+i+2}$, we get by setting $v = (\mu\Lambda' + \Lambda)^{-1} u$

$$0 = ((\mu\Lambda' + T'(u))\, v, (\mu\Lambda' + \Lambda)\, v)_k \geqq c|v|_{k+i}^2 .$$

It follows that $v = 0$, hence also $u = 0$ and the proof of invertibility of the operator $\mu\Lambda' + T'(u)$ is complete.

With the help of (6) we get

$$((\mu\Lambda' + T'(u))\, v, (\mu\Lambda' + \Lambda)\, v)_k \leqq \mu|\Lambda' v|_k \, |(\mu\Lambda' + T'(u))\, v|_k +$$
$$+ c\|(\mu\Lambda' + T'(u))\, v\|_k \, |v|_{k+i} \leqq c\|(\mu\Lambda' + T'(u))\, v\|_k \, (\mu|\Lambda' v|_k + |v|_{k+i})$$

and this inequality together with (15) and (8) yields (11).

Proof of C. For v satisfying $(\mu\Lambda' + T'(u))\, v = g$, (11) and (7) imply

$$\|T'(u)\, v - g\|_0 = \mu\|\Lambda' v\|_0 \leqq \mu|\Lambda' v|_1 \leqq c\, \mu|v|_{k+i} \leqq c\, \mu\|g\|_k ,$$

$$\|v\|_{k+i} \leqq |v|_{k+i+1} \leqq |v|_{k+i}^{1-\tau} \, |v|_{k+i+2}^{\tau} \leqq c\|g\|_k \, \mu^{-\tau} .$$

Now for $Q > 1$ we take $\mu = Q^{-2}$ and (iv) follows with $\nu = 1/\tau$. The last inequality in (iv) immediately follows from (4), (5), (6) for $l = 0$ and ε sufficiently small.

Remark. In some particular cases the assumptions (4), (5) and (6) are needed only for $k = 0$ and the relation similar to (11) can be derived by the regularisation process, see [2] and [5]. Also the assumption that e_n form a basis is needed only in H_0.

3. PERIODIC SOLUTIONS OF THE EQUATIONS

$$u_{tt} + (-1)^p \frac{\partial^p u}{\partial x^{2p}} + \alpha u_t - u = \varepsilon\, F(u) .$$

First we shall deal with the equation

$$(16) \qquad Lu = u_{tt} - u_{xx} + \alpha u_t - u = \varepsilon f(\cdot, \cdot, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = \varepsilon\, F(u)$$

for $(t, x) \in G = R \times \langle 0, \pi \rangle$, $\alpha > 0$, and look for a solution satisfying (1).

In this case it is not possible to use Moser's theorem directly because of the kernel of the linear operator being nontrivial.

Denote $H_0 = \{u \in L^2(G),\ u(t + 2\pi, x) = u(t, x)\}$. H_0 can be written as a direct sum of two orthogonal subspaces, $H_0 = R(L) \oplus N(L)$, where $R(L)$ is the range of the operator L and $N(L) = \{\lambda \sin x\}$ is the null space of L. Denote by P the projector of H_0 onto $R(L)$. Then the equation (16) is equivalent to the system

$$Lu = \varepsilon PF(u)\,, \quad (I - P)\,F(u) = 0\,.$$

We shall look for the solution in the form $u = v + w$, where $v = Pu$, $w = (I - P)\,u$. It will be proved that for $w \in N(L)$, $|w|_k \leqq R$ and ε sufficiently small there exists $v \in R(L)$ such that

$$Lv = \varepsilon PF(v + w)\,, \tag{17}$$

and then w will be determined from the equation

$$(I - P)\,F(v(w) + w) = 0 \tag{18}$$

or, equivalently, $(F(v(w) + w), w) = 0$. This is equivalent to finding a root of the function $g(\lambda) = \int_0^{2\pi} \int_0^{\pi} F(v(\lambda \sin x) + \lambda \sin x)\,(t, x)\,.\,\sin x \,\mathrm{d}x\,\mathrm{d}t$. The function g is continuous on $\langle -R, R\rangle$ if f is continuous in all its arguments and v satisfying (17) continuously depends on w in the C^2 norm.

The existence of a solution to the equation (17) and its continuous dependence on w can be proved with the help of Theorem 1.

Let H_k be the completion of the space

$$\left\{\varphi \in C^\infty,\ \frac{\partial^{2l}}{\partial x^{2l}}\,\varphi(t, 0) = \frac{\partial^{2l}}{\partial x^{2l}}\,\varphi(t, \pi) = 0,\ l = 0, 1, 2, \ldots\right\}$$

in the norm

$$|\varphi|_k = \sum_{k_1 + k_2 \leqq k} \int_0^{2\pi} \int_0^{\pi} \left|\frac{\partial^{k_1 + k_2}\varphi}{\partial t^{k_1}\,\partial x^{k_2}}\,(t, x)\right|^2 \mathrm{d}x\,\mathrm{d}t$$

and let $\hat{H}_k = \{\varphi \in H_0,\ \varphi \in H_k,\ \varphi_t \in H_k\}$, $\|\varphi\|_k = |\varphi|_k + |\varphi_t|_k$. The system $\{e_{mn}\} = \{\sin mx\,.\,\mathrm{e}^{int}\}_{m=1, n=-\infty}^{\infty}$ is an orthogonal basis in H_k and e_{mn} are eigenvectors of the operator L.

Now the spaces PH_k and the operators L, F and Λ,

$$\Lambda u = -u_{ttt} - \lambda u_{tt} - \mu u_{xx} + \lambda_1 u_t + \mu_1 u \quad \text{with suitable} \quad \lambda, \mu, \lambda_1, \mu_1\,,$$

satisfy the assumptions of Theorem 1. Indeed, using the boundary and periodicity conditions and integrating by parts we get

$$(Lu, \Lambda u)_k = (\alpha - \lambda)\,|u_{tt}|_k^2 + \mu|u_{xx}|_k^2 + (\lambda - \mu)\,|u_{tx}|_k^2 +$$
$$+ (\alpha\lambda_1 - \lambda - \mu_1)\,|u_t|_k^2 + (\mu_1 - \mu)\,|u_x|_k^2 - \mu_1|u|_k^2 \geqq c|u|_{k+2}^2\,.$$

The last inequality follows if we set e.g. $\lambda = \alpha/2$, $\mu = \alpha/10$, $\mu_1 = \alpha/4$, $\lambda_1 = 1$ and use the relation $|u|_k^2 \leqq \frac{1}{2}(|u_t|_k^2 + |u_x|_k^2)$, which holds for $u \in PH_{k+1}$, as $u = \sum u_{mn} \sin mx \,.\, e^{int}$ and $u_{10} = 0$.

To prove (5), we write $F'(u)\, v = \sum a_{ij} v_{t^i x^j}$, $i \geqq 0$, $j \geqq 0$, $i + j \leqq 2$, where $a_{ij} = \partial^{i+j} f / \partial u_{t^i x^j} \in \hat{H}_k$ for $u \in \hat{H}_{k+2}$ and $\|a_{ij}\|_k \leqq MK$ if $\|u\|_{k+2} \leqq K$. This is a consequence of the composition of functions inequality, see [2].

We have to estimate $(F'(u)\, v, \Lambda v)_k$. The most difficult term is $(a_{02} v_{xx}, v_{ttt})_k$. Writing a in place of a_{02} we obtain

$$(a v_{xx}, v_{ttt})_k = -(a_t v_{xx}, v_{tt})_k - (a v_{xxt}, v_{tt})_k \,.$$

As H_k are Banach algebras for $k \geqq 2$, we have all the terms on the right-hand side except for $(a D^k v_{xxt}, D^k v_{tt})$ (D^k denotes the derivative of the order k) estimated by $\|a\|_k \, |v|_{k+2}^2$. Nevertheless,

$$(a D^k v_{xxt}, D^k v_{tt}) = -(a_x D^k v_{xt}, D^k v_{tt}) - (a, (\tfrac{1}{2}(D^k v_{xt})^2)_t) =$$
$$= (a_t, (D^k v_{xt})^2) - (a_x D^k v_{xt}, D^k v_{tt}) \,,$$

which can be estimated in the same way. The other terms in $(F'(u)\, v, \Lambda v)_k$ can be handled analogously.

Integrating by parts and using the periodicity conditions we obtain the estimate (6).

It remains to verify the assumptions (i), (ii), (iii) and (v) of Moser's theorem to get a solution of the equation (17) for each $w = \lambda \sin x$, $|\lambda| \leqq R$. We take $D = \{u \in P\hat{H}_{k+2}, \|u\|_4 \leqq 2R\}$. The assumptions (i), (ii) are now satisfied because of the composition of functions inequality. The left-hand side of (iii) is the norm of a quadratic form in v and its derivatives up to the second order, which can be estimated by the Nirenberg inequality by $c\|v\|_0^{2-5/(k+2)} \|v\|_{k+2}^{5/(k+2)}$ (see [3] and [2]). The Nirenberg inequality gives also in (2) $\tau = \frac{1}{2}$ and $\sigma = \frac{1}{4}k$. In (v) $\nu = 2$ and it can be easily seen that this inequality can be fulfilled if k is large enough.

The assumptions of Moser's theorem are fulfilled and the equation (17) has a solution $v(w) \in D$ for all $w(t, x) = \lambda \sin x$, $|\lambda| \leqq R$.

We shall now prove continuous dependence of v on λ in H_4 with the help of the operator Λ satisfying (4), (5), (6). Let $w_1 = \lambda_1 \sin x$, $w_2 = \lambda_2 \sin x$, $|\lambda_i| \leqq R$ and let v_i be the corresponding solutions of the equation $Lv_i = \varepsilon PF(v_i + w_i)$. Then

$$L(v_1 - v_2) = \varepsilon P(F(v_1 + w_1) - F(v_2 + w_2))$$

and for $v = v_1 - v_2$, $w = w_1 - w_2$ and $u_t = t(v_1 + w_1) + (1 - t)(v_2 + w_2)$ we get

$$c|v|_4^2 \leqq (Lv, \Lambda v)_2 = \varepsilon(P(F(v_1 + w_1) - F(v_2 + w_2)), \Lambda v)_2 =$$
$$= \varepsilon \left(\int_0^1 F'(u_t)(v + w)\, \mathrm{d}t, \Lambda v \right)_2 = \varepsilon \int_0^1 [(F'(u_t)\, v, \Lambda v)_2 + (F'(u_t)\, w, \Lambda v)_2]\, \mathrm{d}t \leqq$$
$$\leqq \varepsilon\, c|v|_4^2 + \varepsilon c \sup_{0 \leqq t \leqq 1} \|F'(u_t)\, w\|_2 \, |v|_4 \,.$$

This implies $|v|_4 \leqq \varepsilon c\|w\|_4 \leqq \varepsilon c|\lambda_1 - \lambda_2|$.

It follows from this relation that the function g defined by (19) is continuous and moreover, if there are λ_1, λ_2 such that $|\lambda_i| \leqq R$ and $(F(\lambda_1 \sin x), \sin x) > 0$, $(F(\lambda_2 \sin x), \sin x) < 0$, then $g(\lambda_1) > 0 > g(\lambda_2)$ if ε is small enough. In this case there exists λ_0 such that $g(\lambda_0) = 0$ and $u = v(\lambda_0) + \lambda_0 \sin x$ is the solution of the problem (16), (1). Thus, we have proved

Theorem 2. *Let F map H_{k+2} into H_k and let λ_1, λ_2 be such that*

$$\int_0^{2\pi}\int_0^{\pi} F(\lambda_1 \sin x) \sin x \,\mathrm{d}x\,\mathrm{d}t > 0\,, \quad \int_0^{2\pi}\int_0^{\pi} F(\lambda_2 \sin x) \sin x \,\mathrm{d}x\,\mathrm{d}t < 0\,. \tag{20}$$

Then, if f is sufficiently smooth and ε is sufficiently small, the problem (16), (1) *possesses a classical solution.*

The same procedure can be used if we treat the equations

$$L_p u = \varepsilon F(u)\,, \quad p = 2, 3, \ldots\,, \tag{21}$$

where

$$L_p u = u_{tt} + (-1)^p \frac{\partial^{2p} u}{\partial x^{2p}} + \alpha u_t - u$$

and

$$F(u) = f(\cdot, \cdot, u, u_t, u_x, u_{tx}, u_{xx}, \ldots, u_{x^p}, u_{tt}) \tag{22}$$

or

$$F(u) = f(\cdot, \cdot, u, u_t, u_x, u_{tx}, u_{xx}, \ldots, u_{x^p}, u_{x^{2p}})\,. \tag{23}$$

The spaces H_k now consist of functions with k derivatives with respect to t and pk derivatives with respect to x. The operator Λ is given by

$$\Lambda u = -u_{ttt} - \lambda u_{tt} + \mu(-1)^p u_{x^{2p}} + \lambda_1 u_t + \mu_1 u \quad \text{if } F \text{ has the form (22)}\,,$$

$$\Lambda u = (-1)^p u_{tx^{2p}} - \lambda u_{tt} + \mu(-1)^p u_{x^{2p}} + \lambda_1 u_t + \mu_1 u \quad \text{if } F \text{ has the form (23)}\,.$$

Theorem 3. *Let F map H_{k+2} into H_k and let λ_1, λ_2 be such that* (20) *is satisfied. Then if f given by* (22) *or* (23) *is sufficiently smooth and ε sufficiently small, the problem* (21), (1) *possesses a classical solution.*

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