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RELATIONAL CHARACTERIZATIONS OF PERMUTABLE AND n-PERMUTABLE VARIETIES

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The first attempt to characterize permutable varieties by relational conditions via Mal'cev conditions was done by H. Werner [5]. This method can be enlarged also for other binary relations than in [5] and it can be used also for n-permutable varieties.

Let $\mathfrak{A} = (A, F)$ be an algebra and R be a binary relation on A. R is said to have the *Substitution Property* on $\mathfrak{A}$ (see [2]) if R is a subalgebra of the direct product $\mathfrak{A} \times \mathfrak{A}$, i.e. if for each n-ary $f \in F$ we have

$$\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in R$$

whenever $\langle a_i, b_i \rangle \in R$ for $i = 1, \ldots, n$. For the sake of brevity, call R *compatible* (with $\mathfrak{A}$) provided R has the Substitution Property on $\mathfrak{A}$.

Denote by Con $(\mathfrak{A})$ the congruence lattice of $\mathfrak{A}$ and by $\vee$ the join in Con $(\mathfrak{A})$; by $\cdot$ is denoted the relational product.

A variety $\mathscr{V}$ is called *n-permutable* (see [1]), if

$$\Theta \vee \Phi = \Theta \cdot \Phi \cdot \Theta \cdot \Phi \ldots \quad (n \text{ factors})$$

for every $\Theta, \Phi \in$ Con $(\mathfrak{A})$ for each $\mathfrak{A} \in \mathscr{V}$. $\mathscr{V}$ is called *permutable* (see [3], [5]) if it is 2-permutable, i.e. if, equivalently,

$$\Theta \cdot \Phi = \Phi \cdot \Theta$$

for every $\Theta, \Phi \in$ Con $(\mathfrak{A})$ and each $\mathfrak{A} \in \mathscr{V}$.

Theorem 1. *For a variety $\mathscr{V}$ of algebras the following conditions are equivalent:*

(1) *for each $\mathfrak{A} \in \mathscr{V}$, every reflexive and transitive binary relation compatible with $\mathfrak{A}$ is a congruence on $\mathfrak{A}$;*

(2) *there exist $n \geqq 1$ and ternary polynomials $p_1, \ldots, p_n$ over $\mathscr{V}$ such that*

$$y = p_1(x, x, y), \quad x = p_n(y, x, y) \quad \textit{and}$$

$$p_i(y, x, y) = p_{i+1}(x, x, y) \quad \textit{for} \quad i = 1, \ldots, n-1;$$

(3) $\mathscr{V}$ *is n-permutable.*

Proof. The equivalence of (2) and (3) is proved in [1].

(1) ⇒ (2): Let $\mathfrak{A} = (A, F)$ be an algebra. The set of all reflexive and transitive compatible relations on $\mathfrak{A}$ forms an algebraic lattice with respect to the set inclusion (see [4]). Hence, there exists the least reflexive and transitive compatible relation on $\mathfrak{A}$ containing a given pair $\langle a, b\rangle$ of A; denote it by $Q(a, b)$.

Prove the following proposition (see also [4]):

(P) $\langle x, y\rangle \in Q(a, b)$ *if and only if there exist unary algebraic functions* $\tau_1, \ldots, \tau_n$ *over* $\mathfrak{A}$ *such that* $x = \tau_1(a)$, $y = \tau_n(b)$ *and* $\tau_i(a) = \tau_{i+1}(b)$ *for* $i = 1, \ldots, n-1$.

Let R be the set of all pairs $\langle x, y\rangle$ such that there exist $\tau_1, \ldots, \tau_n$ fulfilling the second condition of **(P)**. Clearly, R is reflexive and transitive. The Substitution Property of R can be shown easy in the same way as for principal congruences $\Theta(a, b)$ in the Mal'cev Lemma [3]. Since $\langle a, b\rangle \in R$, we conclude $Q(a, b) \subseteq R$. Thus R is a reflexive and transitive compatible relation containing $Q(a, b)$. The inclusion $Q(a, b) \supseteq R$ is trivial, thus **(P)** is proved.

Now, suppose $\mathscr{V}$ is a variety of algebras satisfying (1) and $F_2(x, y)$ is a $\mathscr{V}$-free algebra with the generating set $\{x, y\}$. By (1), $Q(x, y)$ is a congruence on $F_2(x, y)$, i.e. by the symmetry of $Q(x, y)$, also $\langle y, x\rangle \in Q(x, y)$. By **(P)**, there exist unary algebraic functions $\tau_1, \ldots, \tau_n$ such that

$$(*)\qquad \begin{aligned} y &= \tau_1(x) \\ \tau_1(y) &= \tau_2(x) \\ &\vdots \ \ldots \\ \tau_{n-1}(y) &= \tau_n(x) \\ \tau_n(y) &= x\,. \end{aligned}$$

Since τ_i are unary algebraic functions over $F_2(x, y)$, there exist ternary polynomials $p_1, \ldots, p_n$ over $\mathscr{V}$ such that

$$\tau_i(\xi) = p_i(\xi, x, y) \quad \text{for} \quad i = 1, \ldots, n\,.$$

Putting p_i into $(*)$, we obtain (2).

(2) ⇒ (1): Let $\mathscr{V}$ be a variety satisfying (2), $\mathfrak{A} \in \mathscr{V}$ and Q be an arbitrary reflexive and transitive relation compatible with $\mathfrak{A}$. In order to obtain (1), it remains to prove the symmetry of Q. Let $\langle x, y\rangle \in Q$. By the reflexivity and the Substitution Property of R, it follows

$$\begin{aligned} \langle y, p_1(y, x, y)\rangle &= \langle p_1(x, x, y),\ p_1(y, x, y)\rangle \in Q \\ \langle p_1(y, x, y),\ p_2(y, x, y)\rangle &= \langle p_2(x, x, y),\ p_2(y, x, y)\rangle \in Q \\ \vdots \ldots \qquad & \qquad \ldots \qquad\qquad \ldots \\ \langle p_{n-1}(y, x, y), x\rangle &= \langle p_n(x, x, y),\ p_n(y, x, y)\rangle \in Q\,. \end{aligned}$$

The transitivity of Q implies immediately $\langle y, x\rangle \in Q$. Q.E.D.

Theorem 2. *Let $\mathscr{V}$ be a variety of algebras. The following conditions are equivalent*:

(1) *for each $\mathfrak{A} \in \mathscr{V}$, every reflexive and compatible relation on $\mathfrak{A}$ is symmetric*;

(2) *for each $\mathfrak{A} \in \mathscr{V}$, every reflexive and compatible relation on $\mathfrak{A}$ is transitive*;

(3) *for each $\mathfrak{A} \in \mathscr{V}$, every reflexive and compatible relation on $\mathfrak{A}$ is a congruence on $\mathfrak{A}$*;

(4) *for each $\mathfrak{A} \in \mathscr{V}$, every reflexive and symmetric compatible relation on $\mathfrak{A}$ is a congruence on $\mathfrak{A}$*;

(5) *there exists a ternary polynomial t over $\mathscr{V}$ such that $t(x, y, y) = x$, $t(x, x, y) = y$*;

(6) *$\mathscr{V}$ is permutable.*

Proof. The equivalence of (1), (2), (3) and (6) was proved by H. Werner in [5] and the equivalence of (5) and (6) was proved by A. I. Mal'cev in [3]; t is called a *Mal'cev polynomial.* The implication (3) $\Rightarrow$ (4) is evident by (1). In order to prove Theorem 2, it sufficies to show (4) $\Rightarrow$ (5). For this, it is sufficient to prove that in a $\mathscr{V}$-free algebra $F_3(x, y, z)$ with the generating set $\{x, y, z\}$, the pairs $\langle x, y\rangle$, $\langle y, x\rangle$, $\langle y, z\rangle$, $\langle z, y\rangle$, $\langle x, x\rangle$, $\langle y, y\rangle$, $\langle z, z\rangle$ generate the pair $\langle x, z\rangle$. This implies the existence of a 7-ary polynomial p over $\mathscr{V}$ with

$$x = p(x, y, y, z, x, y, z)$$
$$z = p(y, x, z, y, x, y, z)\,.$$

However, $t(x, y, z) = p(x, y, z, y, x, y, z)$ is clearly a Mal'cev polynomial. Q.E.D.

Remark. Other relational conditions among binary compatible relations with combinations of properties reflexivity, symmetry and transitivity imply the triviality of $\mathscr{V}$. One example is shown by the following:

Theorem 3. *For a variety $\mathscr{V}$ of algebras the following conditions are equivalent*:

(1) *for each $\mathfrak{A} \in \mathscr{V}$, every symmetric compatible relation on $\mathfrak{A}$ is transitive*;

(2) *$\mathscr{V}$ contains only one element algebras.*

Proof. Let $\mathfrak{A} = (A, F)$ be an algebra and $a, b, c, d \in A$. The set of all symmetric compatible relations on $\mathfrak{A}$ forms clearly a complete lattice with respect to the set inclusion. Hence there exists the least symmetric compatible relation on $\mathfrak{A}$ containing given pairs $\langle a, b\rangle$, $\langle c, d\rangle$; denote it by $S(a, b, c, d)$. Analogously as in the proof of Theorem 1 it can be proved easily the following proposition:

(Q) $\langle x, y\rangle \in S(a, b, c, d)$ *if and only if there exists a 4-ary polynomial q over $\mathfrak{A}$ such that $x = q(a, b, c, d)$, $y = q(b, a, c, d)$.*

Now, we are ready to prove (1) $\Rightarrow$ (2): Let $F_3(x, y, z)$ be a $\mathscr{V}$-free algebra with the generating set $\{x, y, z\}$, where $\mathscr{V}$ be a variety satisfying (1). Accordingly, $S(x, y, y, z)$ is transitive, i.e. $\langle x, z\rangle \in S(x, y, y, z)$.

By (Q) there exists a 4-ary polynomial q over $\mathscr{V}$ fulfilling

$$x = q(x, y, y, z), \quad z = q(y, x, z, y).$$

Applying the first and then the second identity, we obtain

$$x = q(x, y, y, x) = q(x, z, z, x) = z.$$

Since x, z are free generators, the identity $x = z$ implies that $\mathscr{V}$ contains one element algebras only. Q.E.D.

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