Czechoslovak Mathematical Journal

Josef Niederle

Decomposability conditions for compatible relations

Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 4, 522–524

Persistent URL: http://dml.cz/dmlcz/101908

Czechoslovak Mathematical Journal, 33 (108) 1983, Praha

DECOMPOSABILITY CONDITIONS FOR COMPATIBLE RELATIONS

JOSEF NIEDERLE, Brno

(Received May 18, 1980)

Let $\mathscr{V}$ be a variety of algebras, $\mathscr{R}_A$ a set of compatible relations on any algebra A from $\mathscr{V}$ such that whenever $p : A \to B$ is an onto-homomorphism of algebras from $\mathscr{V}$ then $R \in \mathscr{R}_A$ implies $(p \times p)(R) \in \mathscr{R}_B$. The members of the system $\mathscr{R}$ will be called *$\mathscr{R}$-relations*. For $\mathscr{R}$ may be chosen compatible relations, compatible reflexive relations, compatible symmetric relations, compatible tolerance relations etc.

Let A, B be algebras from $\mathscr{V}$, R_A a relation on A and R_B a relation on B. The relation $\{[[a, b], [a', b']] \mid [a, a'] \in R_A \text{ and } [b, b'] \in R_B\}$ on $A \times B$ will be called the *$\otimes$-product* of R_A and R_B and denoted by $R_A \otimes R_B$. A variety of algebras $\mathscr{V}$ is said to have *decomposable $\mathscr{R}$-relations* if any $\mathscr{R}$-relation on the direct product $A \times B$ of arbitrary algebras A, B from $\mathscr{V}$ is $\otimes$-product of suitable $\mathscr{R}$-relations on A and B.

Proposition. *A variety of algebras $\mathscr{V}$ has decomposable $\mathscr{R}$-relations iff for every pair of algebras A, B from $\mathscr{V}$ and for every $\mathscr{R}$-relation R on $A \times B$ the following holds:*

(i) $[a, b] R [c, d]$ *and* $[a', b'] R [c', d'] \Rightarrow [a, b'] R [c, d']$.

Proof. $\Rightarrow$: Clear.

$\Leftarrow$: Denote $R_A = (p_A \times p_A)(R)$, $R_B = (p_B \times p_B)(R)$, where p_A and p_B are the natural projections. Obviously R_A and R_B are $\mathscr{R}$-relations. Prove $R = R_A \otimes R_B$: : $[[x, y], [x', y']] \in R$ implies $[x, x'] \in R_A$, $[y, y'] \in R_B$ and so $[[x, y], [x', y']] \in \in R_A \otimes R_B$. So $R \subseteq R_A \otimes R_B$. Conversely, $[[x, y], [x', y']] \in R_A \otimes R_B$ implies $[x, x'] \in R_A$, $[y, y'] \in R_B$, and so there exist $\bar{x}, \bar{x}' \in A$ and $\bar{y}, \bar{y}' \in B$ such that $[[x, \bar{y}], [x', \bar{y}']] \in R$ and $[[\bar{x}, y], [\bar{x}', y']] \in R$. By (i) $[[x, y], [x', y']] \in R$. Hence $R_A \otimes R_B \subseteq R$, so $R = R_A \otimes R_B$. Q.E.D.

Example. The variety of all lattices with compatible reflexive relations satisfies (i), since

$$[a, b'] = ([a, b] \wedge [a \vee c,\ b' \wedge d']) \vee ([a', b'] \wedge [a \wedge c,\ b' \vee d'])$$
$$[c, d'] = ([c, d] \wedge [a \vee c,\ b' \wedge d']) \vee ([c', d'] \wedge [a \wedge c,\ b' \vee d']) .$$

Thus the variety of lattices has decomposable reflexive relations and so decomposable tolerances, as stated in [1].

The condition (i) can be rewritten for reflexive relations as

(ii) $f([a, b], [a', b'], [x_1, y_1], \ldots, [x_n, y_n]) = [a, b']$
$f([c, d], [c', d'], [x_1, y_1], \ldots, [x_n, y_n]) = [c, d']$

and for tolerances as

(iii) $f([a, b], [a', b'], [c, d], [c', d'], [x_1, y_1], \ldots, [x_n, y_n]) = [a, b']$
$f([c, d], [c', d'], [a, b], [a', b'], [x_1, y_1], \ldots, [x_n, y_n]) = [c, d']$

where $[x_i, y_i]$ are suitable elements of $A \times B$ and f a $\mathscr{V}$-polynomial.

Aplying these conditions to $F_{\mathscr{V}}(4) \times F_{\mathscr{V}}(4)$ and the compatible reflexive (tolerance) relation generated by $[s, s]\, R[u, u]$ and $[t, t]\, R[v, v]$ one has

(ii)′ $f([s, s], [t, t], [g_1(s, t, u, v), h_1(s, t, u, v)], \ldots, [g_n(s, t, u, v), h_n(s, t, u, v)]) = [s, t]$
$f([u, u], [v, v], [g_1(s, t, u, v), h_1(s, t, u, v)], \ldots, [g_n(s, t, u, v), h_n(s, t, u, v)]) = [u, v]$

and

(iii)′ $f([s, s], [t, t], [u, u], [v, v], [g_1(s, t, u, v), h_1(s, t, u, v)], \ldots, [g_n(s, t, u, v), h_n(s, t, u, v)]) = [s, t]$
$f([u, u], [v, v], [s, s], [t, t], [g_1(s, t, u, v), h_1(s, t, u, v)], \ldots, [g_n(s, t, u, v), h_n(s, t, u, v)]) = [u, v]$

where s, t, u, v are the free generators of $F_{\mathscr{V}}(4)$ and g_i, h_i suitable quaternary $\mathscr{V}$-polynomials. Conversely, a variety satisfying (ii)′ satisfies (ii) and a variety satisfying (iii)′ satisfies (iii), since the primed conditions are in fact systems of identities holding in $\mathscr{V}$.

The above may be summed as follows.

Theorem 1. *A variety of algebras $\mathscr{V}$ has decomposable reflexive relations iff there exist an $(n+2)$-ary $\mathscr{V}$-polynomial f and quaternary $\mathscr{V}$-polynomials $g_1, \ldots, g_n, h_1, \ldots, h_n$ such that*

$$f(s, t, g_1(s, t, u, v), \ldots, g_n(s, t, u, v)) = s$$
$$f(u, v, g_1(s, t, u, v), \ldots, g_n(s, t, u, v)) = u$$
$$f(s, t, h_1(s, t, u, v), \ldots, h_n(s, t, u, v)) = t$$
$$f(u, v, h_1(s, t, u, v), \ldots, h_n(s, t, u, v)) = v$$

are $\mathscr{V}$-identities.

Theorem 2. *A variety of algebras $\mathscr{V}$ has decomposable tolerances iff there exist an $(n+4)$-ary $\mathscr{V}$-polynomial f and quaternary $\mathscr{V}$-polynomials $g_1, \ldots, g_n, h_1, \ldots, h_n$ such that*

$$f(s, t, u, v, g_1(s, t, u, v), \ldots, g_n(s, t, u, v)) = s$$
$$f(u, v, s, t, g_1(s, t, u, v), \ldots, g_n(s, t, u, v)) = u$$
$$f(s, t, u, v, h_1(s, t, u, v), \ldots, h_n(s, t, u, v)) = t$$
$$f(u, v, s, t, h_1(s, t, u, v), \ldots, h_n(s, t, u, v)) = v$$

are $\mathscr{V}$-identities.

Only trivial varieties have decomposable symmetric relations. The same statement holds for any system of relations containing all symmetric relations.

Reference

[1] *J. Niederle:* A note on tolerance lattices of products of lattices. Čas. pěs. mat. *107* (1982), 114—115.

Author's address: 615 00 Brno 15, Viniční 60, ČSSR.