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SOME ASPECTS OF GREEN'S RELATIONS ON PERIODIC SEMIGROUPS

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1. INTRODUCTION

A semigroup S is called *periodic* if to each element a of S there corresponds an idempotent e and a positive integer n such that $a^n = e$; the element a is then said to *belong to e*. In his pioneering study of periodic semigroups, Schwarz [7] introduced the equivalence $\mathscr{K}$, each of whose classes is associated with a particular idempotent e and consists of all elements belonging to e.

Some of the earliest structure theorems for periodic semigroups are due to Schwarz [8] and Yamada [11], both of whom were concerned with the commutative case. Later Petrich [6] introduced the notion of weak commutativity and applied it to periodic semigroups. Sedlock [9] used a generalized form of this concept to characterize certain periodic semigroups as semilattices of more restricted semigroups. His principal hypotheses are in the form of connections between Green's relations and the equivalence $\mathscr{K}$.

The main purpose of this paper is to give a characterization of those periodic semigroups for which Green's relation $\mathscr{J}$ is included in $\mathscr{K}$. Section 2 is a development of properties of such semigroups. Section 3 includes the primary structure theorem for these semigroups, as well as a characterization of periodic semigroups which are $\mathscr{J}$-trivial.

The final section is a brief discussion of the current status of the structure problem for periodic semigroups in which $\mathscr{J} \subseteq \mathscr{K}$.

2. THE EQUIVALENCE $\mathscr{K}$

The notation and terminology of [3] will be followed wherever applicable. In particular, if $\mathscr{T}$ is any one of Green's relations $\mathscr{J}$, $\mathscr{D}$, $\mathscr{L}$, $\mathscr{R}$ or $\mathscr{H}$ on a semigroup S and if x is an element of S, the $\mathscr{T}$-equivalence class containing x will be denoted by T_x. It is well known that these five equivalences form a sublattice of $S \times S$ (partially ordered by inclusion), with greatest element $\mathscr{J}$ and least element $\mathscr{H}$.

Green [4] noted the following property of periodic semigroups.

2.1. Proposition. *If S is a periodic semigroup then $\mathscr{D} = \mathscr{J}$.*

The relation $\mathscr{K}$, introduced by Schwarz [7], has proved to be an extremely useful tool in the study of periodic semigroups.

2.2. Definition. Let S be a periodic semigroup, with set of idempotents E_S. For a, b in S, $a \mathscr{K} b$ if and only if there exist positive integers m, n and an element e in E_S such that $a^m = b^n = e$.

Evidently $\mathscr{K}$ is an equivalence relation on any periodic semigroup S. The $\mathscr{K}$-class containing an element x of S will be denoted by K_x. Schwarz established the following connection between $\mathscr{H}$-classes and $\mathscr{K}$-classes of a periodic semigroup.

2.3. Lemma. (Th. 8 of [7]). *For each idempotent e in a periodic semigroup S,*

$$eK_e = K_e e = H_e .$$

The balance of this section is concerned with the properties of periodic semigroups which satisfy the condition $\mathscr{J} \subseteq \mathscr{K}$. The equivalence $\mathscr{K}$ is, of course, idempotent-separating on every such semigroup.

The following notation will be used extensively. If s and t are elements of a semigroup S and if k is a positive integer, then $[s, t]_k$ is defined by

$$(2.4) \qquad [s, t]_k = (st)^{k-1} s = s(ts)^{k-1} .$$

In particular, $[s, t]_1$ is understood to denote s. Note also that

$$(2.5) \qquad t[s, t]_k = (ts)^k = [t, s]_k s .$$

2.6. Lemma. *Let S be a periodic semigroup with $\mathscr{J} \subseteq \mathscr{K}$ and let $a, b \in S$, $e \in E_S$. Then $ab \in K_e$ if and only if $ba \in K_e$.*

Proof. Suppose that $ab \in K_e$ and $ba \in K_f$, where $f \in E_S$. Thus $(ab)^n = e$ and $(ba)^n = f$ for some positive integer n, so

$$e = (ab)^{2n} = a(ba)^n (ba)^{n-1} b \in S^1 f S^1 .$$

Similarly $f \in S^1 e S^1$ so $e \mathscr{J} f$. Thus $e \mathscr{K} f$ so, since $\mathscr{K}$ is idempotent-separating, $e = f$. □

The following result will be used to show that the $\mathscr{K}$-classes of certain periodic semigroups S are in fact subsemigroups of S.

2.7. Lemma. *Let S be a periodic semigroup with $\mathscr{J} \subseteq \mathscr{K}$ and let $a, b \in S$. Then $ab \mathscr{K} a^i b$ for every positive integer i.*

Proof. Suppose that $ab \in K_e$ and $a^2 b \in K_f$, where $e, f \in E_S$. By Lemma 2.6, $ba \in K_e$, $aba \in K_f$ and $ba^2 \in K_f$. Thus there is a positive integer n such that

(2.8) $$(ab)^n = (ba)^n = e$$

and

(2.9) $$(a^2b)^n = (aba)^n = (ba^2)^n = f.$$

If $n = 1$ then $f = ba^2 = (ba)a = ea$ so $ef = f$. Also $e = e^2 = (ba)(ab) = b(a^2b) = bf$, so $ef = e$. Hence $e = f$.

Suppose that $n > 1$. Since $(ab)^n a = a(ba)^n$ then, by (2.8)

$$ea = ae.$$

Similarly

$$eb = be.$$

Moreover, since $(a^2b)^n a = a(aba)^n$, then

$$fa = af.$$

Similarly $b(a^2b)^n = (ba^2)^n b$, so

$$fb = bf.$$

For each positive integer k, let

$$a_k = [a, b]_k, \quad b_k = [b, a]_k, \quad c_k = [a, ba]_k.$$

By (2.5) and (2.8), $(ba)c_n = (ba^2)^n$, so by (2.9),

(2.10) $$(ba)c_n = f.$$

Moreover, since f commutes with both a and b,

(2.11) $$fc_k = c_kf \quad \text{for all} \quad k \geqq 1.$$

Since

$$e = e^2 = (ba)^n (ab)^n = (ba)^{n-2} b(aba)^2 b(ab)^{n-2},$$

it follows from (2.4) that

$$e = b_{n-1}(aba)^2 b_{n-1}.$$

Assume inductively that, for some positive integer k,

(2.12) $$e = b_{n-1}(aba)^{k+1} b_{n-1}^k.$$

Since e commutes with both a and b, it also commutes with b_k. Thus

$$e = e^2 = b_{n-1}(aba)^{k+1} eb_{n-1}^k =$$

$$= [b_{n-1}(aba)^{k+1}][(aba)b_{n-1}] b_{n-1}^k = b_{n-1}(aba)^{k+2} b_{n-1}^{k+1}.$$

Hence (2.12) holds for every positive integer k.

Set k equal to $n - 1$ in (2.12); by (2.9),

$$e = b_{n-1}fb_{n-1}^{n-1}.$$

Thus $e \in S^1fS^1$.

Furthermore, by (2.4),

$$c_n = [a, ba]_n = (aba)^{n-1} a$$

and similarly

$$c_n = a(ba^2)^{n-1} .$$

Hence, by (2.9),

$$f = f^2 = (aba)^n (ba^2)^n = (aba)^{n-1} a(ba)^2 a(ba^2)^{n-1}$$

so, by (2.4),

$$f = c_n(ba)^2 c_n .$$

Assume inductively that, for some positive integer k,

$$f = c_n(ba)^{k+1} c_n^k . \tag{2.13}$$

Then, by (2.11) and (2.10),

$$f = f^2 = c_n(ba)^{k+1} fc_n^k = c_n(ba)^{k+2} c_n^{k+1} .$$

Therefore (2.13) holds for every positive integer k.

Replace k by $n - 1$ in (2.13) to obtain

$$f = c_n(ba)^n c_n^{n-1} .$$

By (2.8), $f = c_n e c_n^{n-1} \in S^1 e S^1$.

Thus $e \mathscr{J} f$ so $e = f$, and hence $ab \mathscr{H} a^2 b$. By induction, $ab \mathscr{H} a^i b$ for every positive integer i. □

Again let S be a periodic semigroup with $\mathscr{J} \subseteq \mathscr{H}$. Suppose $e \in E_S$ and $a, b \in K_e$, say $a^n = e$. By Lemma 2.7, $ab \mathscr{H} a^n b = eb$. Moreover, by Lemma 2.3, $eb \in eK_e = H_e \subseteq K_e$. Thus $K_{ab} = K_{eb} = K_e$, so K_e is a subsemigroup of S. This establishes the following result:

2.14. Lemma. *Let S be a periodic semigroup with $\mathscr{J} \subseteq \mathscr{H}$ and let $e \in E_S$. Then K_e is a periodic unipotent subsemigroup of S.*

Lemma 2.7 also yields a description of the quotient set $S/\mathscr{H}$.

2.15. Lemma. *Let S be a periodic semigroup with $\mathscr{J} \subseteq \mathscr{H}$. If e, f, g are idempotents of S such that $ef \in K_g$ then*

$$K_e K_f \subseteq K_g .$$

Proof. Let $b \in K_f$; thus $b^n = f$ for some positive integer n. By Lemma 2.7, $be \mathscr{H} b^n e = fe$ so, by Lemma 2.6, $eb \mathscr{H} ef \mathscr{H} g$. Thus $eb \in K_g$ so

$$eK_f \subseteq K_g . \tag{2.16}$$

Now suppose $a \in K_e$ and $b \in K_f$, with $a^m = e$ for some positive integer m. Then $ab \mathscr{H} a^m b$ and, by (2.16)

$$a^m b = eb \in eK_f \subseteq K_g$$

so $ab \in K_g$. □

Lemmas 2.6, 2.14 and 2.15 together yield the following result.

2.17. Theorem. *Let S be a periodic semigroup for which $\mathscr{J} \subseteq \mathscr{H}$. Then S is a semilattice of periodic unipotent semigroups.*

3. STRUCTURE THEOREMS

Petrich [6] introduced the concept of weak commutativity and used it to characterize a class of periodic semigroups.

3.1. Definition. A semigroup S is said to be *weakly commutative* if for each a, b in S there exist x, y in S and a positive integer k such that

$$(ab)^k = xa = by .$$

3.2. Theorem. (6.8 of [6]) *If S is a periodic semigroup, the following are equivalent*:

(i) *For all e, f in E_S, $K_e K_f \subseteq K_{ef} = K_{fe}$.*
(ii) *S is weakly commutative and E_S is commutative.*
(iii) *S is weakly commutative and E_S is a semigroup.*

In his study of Green's relations and the equivalence $\mathscr{H}$ on periodic semigroups, Sedlock [9] used a generalization of weak commutativity to obtain several structure theorems. They are restated here for comparison with the principal result of this section.

3.3. Definition. (3.3 of [9]) A semigroup S is said to be *left [right] weakly commutative* if for each a, b in S there exist an element x of S and a positive integer k such that

$$(ab)^k = bx \ [(ab)^k = xa] .$$

3.4. Theorem. (2.9 of [9]) *The following are equivalent on a periodic semigroup S*:

(i) *S is a semilattice of groups*;
(ii) *S is a union of groups and is weakly commutative*;
(iii) $\mathscr{H} = \mathscr{D}$.

3.5. Theorem. (3.5 of [9]) *The following are equivalent on a periodic semigroup S*:

(i) *S is a semilattice of right groups*;
(ii) *S is a union of groups and is left weakly commutative*;
(iii) $\mathscr{H} = \mathscr{L}$.

3.6. Theorem. (2.3 and 3.8 of [9]) *The following are equivalent ona periodic semigroup S*:

(i) *S is a semilattice of completely simple semigroups*;

(ii) *S is a union of groups*;

(iii) $\mathscr{K} = \mathscr{H}$;

(iv) $\mathscr{K} \subseteq \mathscr{J}$.

Theorem 2.17 can be recast into a form analogous to that of Sedlock's theorems.

3.7. Theorem. *The following are equivalent on a periodic semigroup S*:

(i) *S is a semilattice of unipotent semigroups*;

(ii) *S is a union of unipotent semigroups and is weakly commutative*;

(iii) $\mathscr{J} \subseteq \mathscr{K}$.

Proof. Assume (i). If $a, b \in S$ then $ab, ba \in T$ for some unipotent subsemigroup T of S, say with idempotent e. Moreover, $(ab)^n = (ba)^n = e$ for some positive integer n, so S is weakly commutative. Thus (ii) is valid.

By Proposition 2.6 of [9], $\mathscr{J} \subseteq \mathscr{K}$ in every weakly commutative periodic semigroup, so (ii) implies (iii). Finally, by Theorem 2.17, (iii) implies (i). □

A semigroup on which Green's relation $\mathscr{J}$ is the identity is said to be *$\mathscr{J}$-trivial*; such a semigroup is necessarily combinatorial.

These semigroups arise in formal language theory; for example, Simon [10] has shown that a language is piecewise testable if and only if its syntactic monoid is $\mathscr{J}$-trivial.[1]) Theorem 3.7 yields a global description of periodic $\mathscr{J}$-trivial semigroups.

A semigroup with zero is said to be *nil* if each of its elements is nilpotent; in particular, every nil semigroup is periodic.

3.8. Lemma. *Let S be a semilattice of nil semigroups. If* $xy \in K_e$ *for some* x, y *in S and some e in* E_S, *then there is a positive integer k such that* $[x, y]_k = e$.

Proof. Note that the nil components of S are just the $\mathscr{K}$-classes K_e of S, where e ranges over E_S.

Suppose that $x \in K_f$, $y \in K_g$ and $xyx \in K_h$, where $f, g, h \in E_S$. Since $xy \in K_e$ then $yx \in K_e$ so $x(yx) \in K_fK_e \cap K_h$. Thus $K_fK_e \cap K_h \neq \emptyset$ so $K_fK_e \subseteq K_h$. Therefore $(K_fK_e) K_e \subseteq K_hK_e \cap K_h$, i.e., $K_hK_e \subseteq K_h$.

Moreover, $(xy)^2 \in K_e$ and $(xyx)\, y \in K_hK_g$, so $K_hK_g \subseteq K_e$. Thus, as above, $K_hK_e \subseteq$ $\subseteq K_e$. Since also $K_hK_e \subseteq K_h$ then $h = e$, so $xyx \in K_e$.

If k is any integer exceeding 2 then, by (2.4),

$$[x. y]_k = (xyx)(yx)^{k-2} \in K_eK_e \subseteq K_e .$$

Since $xy \in K_e$ then $(xy)^n = (yx)^n = e$ for some positive integer n, assumed to be greater than 2. Therefore

$$[x, y]_{2n+1} = (xy)^{2n} x = ex = (xy)^n x = [x, y]_{n+1} .$$

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[1]) For an account of this and related results, see Lallement [5].

Thus, since $[x, y]_{n+1} \in K_e$,

$$[x, y]_{n+1} = [x, y]_{2n+1} = [x, y]_{n+1} (yx)^n = [x, y]_{n+1} e = e. \quad \square$$

3.9. Theorem. *The following are equivalent on a periodic combinatorial semigroup S:*

(i) *S is a semilattice of nil semigroups;*

(ii) *S is a union of nil semigroups and is weakly commutative;*

(iii) *S is $\mathcal{J}$-trivial.*

Proof. A periodic nil semigroup is necessarily unipotent, and a unipotent subsemigroup of a periodic combinatorial semigroup must be nil. Hence the equivalence of conditions (i) and (ii) follows from Theorem 3.7.

Suppose that S is $\mathcal{J}$-trivial. Then $\mathcal{J} \subseteq \mathcal{K}$ so, again by Theorem 3.7, S is a semilattice of nil semigroups.

Finally, suppose that S is a semilattice of its $\mathcal{K}$-classes. If a, b in S satisfy $a \mathcal{L} b$ then there exist x, y in S^1 such that $xa = b$, $yb = a$. Suppose that xy, and hence also yx, lies in K_e, with e in E_S. Then $(yx) a = y(xa) = yb = a$, so

$$[x, y]_k a = x(yx)^{k-1} a = xa = b, \quad \text{all} \quad k \geqq 1.$$

Thus, by Lemma 3.8, $ea = b$. Dually $eb = a$, so

$$a = eb = e(eb) = ea = b.$$

Therefore S is $\mathcal{L}$-trivial; similarly S is $\mathcal{R}$-trivial and hence $\mathcal{D}$-trivial. And the periodicity of S implies that $\mathcal{J} = \mathcal{D}$. $\square$

3.10. Corollary. *A periodic $\mathcal{J}$-trivial semigroup is a semilattice of nil semigroups.*

4. CONCLUDING REMARKS

In view of Theorem 3.7 it is natural to look more closely at periodic unipotent semigroups. Let S be such a semigroup, with idempotent e. Let $x \in S$ and let G_x be the maximum subgroup (and minimum ideal) of the cyclic subsemigroup $\langle x \rangle$ of S. Thus $G_x \subseteq H_e$, the maximum subgroup of S, so

$$xe \in xG_x \subseteq G_x \subseteq H_e.$$

Therefore

$$xH_e = x(eH_e) = (xe) H_e \subseteq H_e.$$

Similarly $H_e x \subseteq H_e$, so H_e is an ideal of S.

Moreover, $x^n = e$ for some positive integer n, so the Rees quotient semigroup S/H_e is nil. Therefore S is an extension of the periodic group H_e by the nil semigroup S/H_e.

Conversely if a semigroup S is an extension of a periodic group G by a nil semigroup N then to each x in S corresponds a positive integer n such that $x^n \in G$. Thus $(x^n)^m$ is equal to e, the identity element of G, for some positive integer m, so S is periodic. In particular if x is idempotent then $x = e$, so S is unipotent.

These remarks are summarized as follows:

4.1. Theorem. *Every periodic unipotent semigroup is an extension of a periodic group by a nil semigroup and conversely.*

It is well known (see, e.g., Theorem 4.19 of [3]) that all of the extensions of a semigroup S by a semigroup $T = T^0$ are determined by the partial homomorphisms of the partial groupoid $T^* = T \setminus 0$ into S. Moreover, these partial homomorphisms have been characterized by Arendt and Stuth [2].

It follows from Theorems 3.7 and 4.1 that an adequate description of all periodic semigroups for which $\mathscr{J} \subseteq \mathscr{H}$ depends upon a determination of three classes: the class of periodic groups, the class of nil semigroups, and, for a given semilattice Y and set $\{S_\gamma : \gamma \in Y\}$ of periodic unipotent semigroups, the class of semilattices Y of the S_γ. A partial solution for the latter problem, in the commutative case, has been given by Arendt and Stuth [1].

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