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Bohdan Zelinka

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HOMOMORPHISMS OF INFINITE BIPARTITE GRAPHS ONTO COMPLETE BIPARTITE GRAPHS

BOHDAN ZELINKA, Liberec

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Let B be a bipartite graph on the vertex sets C, D. A bicomplete homomorphism of B onto a complete bipartite graph $K_{r,s}$ is a homomorphism φ of B onto $K_{r,s}$ such that $\varphi(x) = \varphi(y)$ only if either both x, y belong to C, or both x, y belong to D.

F. Harary, D. Hsu and Z. Miller [1] have defined the bichromaticity $\beta(B)$ of a connected bipartite graph B as the maximum value of $r + s$ for all complete bipartite graphs $K_{r,s}$ onto which B can be mapped by a bicomplete homomorphism. They have considered only finite graphs. For an infinite bipartite graph B it can be easily proved that its bichromaticity is equal to the cardinality of its vertex set. But we shall introduce a similar concept which seems to be interesting also for infinite graphs.

For a connected bipartite graph B we define $\beta_0(B)$ as the supremum of all values of $\min(r, s)$ for all complete bipartite graphs $K_{r,s}$ onto which B can be mapped by a bicomplete homomorphism; the numbers r, s may be finite or infinite.

A matching [2] of a bipartite graph B is an arbitrary set of edges of B, no two of which have a common end vertex. (In [2] the definition is slightly different from this one, but the difference is not essential and for our purposes the above definition is more convenient.)

We shall prove two theorems.

Theorem 1. *Let B be a connected bipartite graph on the sets C, D, let $|C| \geqq \aleph_0$, $|D| \geqq \aleph_0$. Then $\beta_0(B)$ is less than or equal to the supremum of cardinalities of all matchings of B. If there exists at least one infinite matching of B, then $\beta_0(B)$ is equal to this supremum.*

Proof. If a bipartite graph B is mapped by a bicomplete homomorphism onto a complete bipartite graph B_0 and B_0 has a matching M_0 of the cardinality $\mathfrak{n}$, then evidently B has a matching of the cardinality $\mathfrak{n}$ (the end vertices of edges of M are vertices whose images are the end vertices of edges of M_0). If $\beta_0(B) = \mathfrak{n}$, then there exists a bicomplete homomorphism of B onto a complete bipartite graph B_0 on the vertex sets C_0, D_0 such that $|C_0| \geqq |D_0| = \mathfrak{n}$. The graph B_0 has a matching of the cardinality $\mathfrak{n}$, hence so has B and thus $\beta_0(B)$ is less than or equal to the supremum of cardinalities of matchings of B. Now suppose that there exists a matching M of B of an infinite cardinality $\mathfrak{n}$. The edges of M can be denoted by e_ι, where ι runs through the set $\Omega(\mathfrak{n})$ of all ordinal numbers which are less than the least ordinal number of the cardinality $\mathfrak{n}$. (This assertion follows from Axiom of Choice.) If $\iota \in \Omega(\mathfrak{n})$, then let $u(\iota)$ (or $v(\iota)$) be the end vertex of e_ι which belongs to C (or D, respectively). Let $C^* = \{u(\iota) \mid \iota \in \Omega(\mathfrak{n})\}$, $D^* = \{v(\iota) \mid \iota \in \Omega(\mathfrak{n})\}$. Let $\mathscr{P}$ be a partition of $\Omega(\mathfrak{n})$ into $\mathfrak{n}$ classes, each of which has the cardinality $\mathfrak{n}$. Let the classes of $\mathscr{P}$ be $P(\iota)$ for all $\iota \in \Omega(\mathfrak{n})$. The elements of each $P(\iota)$ can be denoted by $a(\iota, \varkappa)$ for all $\varkappa \in \Omega(\mathfrak{n})$. Now for each $\varkappa \in \Omega(\mathfrak{n})$ let $Q(\varkappa) = \{a(\iota, \varkappa) \mid \iota \in \Omega(\mathfrak{n})\}$. Then $\mathscr{Q} = \{Q(\varkappa) \mid \varkappa \in \Omega(\mathfrak{n})\}$ is a partition of $\Omega(\mathfrak{n})$ into $\mathfrak{n}$ classes, each of which has the cardinality $\mathfrak{n}$ and has non-empty intersections with all classes of $\mathscr{P}$. Then there exists a bicomplete homomorphism φ of B onto a complete bipartite graph B_0 on the sets C_0, D_0 such that $|C_0| = |D_0| = \mathfrak{n}$. This mapping is constructed so that $\varphi(x) = \varphi(y)$ if and only if either $x = u(\iota_1)$, $y = u(\iota_2)$ and ι_1, ι_2 belong to the same class of $\mathscr{P}$, or $x = v(\varkappa_1)$, $y = v(\varkappa_2)$ and $\varkappa_1$, $\varkappa_2$ belong to the same class of $\mathscr{Q}$. Therefore $\beta_0(B) \geqq \mathfrak{n}$. We have proved that $\beta_0(B)$ is greater than or equal to the cardinalities of all matchings of B. As it cannot be greater than their supremum, it is equal to it.

We shall give an example showing that in the case when all matchings of B are finite $\beta_0(B)$ need not be equal to the mentioned supremum. Let n be an arbitrary positive integer greater than 1. Take a path P of the length $2n - 2$ and two disjoint infinite sets R, S of vertices not belonging to P. Join all vertices of R with one terminal vertex of P and all vertices of S with the other. The graph B thus obtained is a bipartite graph on infinite vertex sets C, D. It has a matching of the cardinality n. But evidently $\beta_0(B) < n$.

Theorem 2. *Let B be a connected bipartite graph on the sets C, D such that $|C| = \mathfrak{a} \geqq \aleph_0$, $|D| < \aleph_0$. Let D^* be the set of vertices of D which have the degree $\mathfrak{a}$. Then $\beta_0(B) \geqq |D^*|$.*

Proof. Let $|D^*| = b$. Denote the vertices of D^* by $v_1, \ldots, v_b$. For each $i = 1, \ldots, b$ let M_i be the set of vertices which are adjacent to v_i in B. We shall define recurrently the sets $P_1, \ldots, P_b$ and the graphs $B_1, \ldots, B_b$. Put $P_1 = M_1$, $B_1 = B$. Now suppose that we have defined the set P_i and the graph B_i for some $i \leqq b - 1$. Further suppose that the vertices $v_1, \ldots, v_b$ are vertices of B_i and have the degree $\mathfrak{a}$ in it. Let M_{i+1}^i be the set of vertices which are adjacent to v_{i+1} in B_i. If $|P_i \cap M_{i+1}^i| = \mathfrak{a}$, we put $P_{i+1} = P_i \cap M_{i+1}^i$. If not, then the sets $P_i - M_{i+1}^i$, $M_{i+1}^i - P_i$ have the cardinality $\mathfrak{a}$. We choose a one-to-one correspondence between these sets and identify the pairs of corresponding vertices. The union of $P_i \cap M_{i+1}^i$ and the set of vertices obtained by this identification will be P_{i+1} and the graph obtained from B_i by this identification will be B_{i+1}. As we have always identified only pairs of vertices, the degrees of vertices $v_1, \ldots, v_b$ remain to be equal to $\mathfrak{a}$. After a finite number of steps we obtain the graph B_b in which there exists a set of vertices of the cardinality $\mathfrak{a}$ which are adjacent to all vertices $v_1, \ldots, v_b$; therefore we have a subgraph of B_b

which is a complete bipartite graph, one of whose vertex sets has the cardinality $\mathfrak{a}$ and the other the cardinality b. Evidently the graph B can be mapped onto this graph by a bicomplete homomorphism and $\beta_0(B) \geqq b = |D^*|$.

Note that be $\beta_0(B) > |D^*|$ may occur. Let b be an arbitrary positive integer. Take the complete bipartite graph $K_{b,b}$, choose a vertex u in it, add a set R of vertices (not belonging to $K_{b,b}$) of the cardinality $\mathfrak{a} \geqq \aleph_0$ to $K_{b,b}$ and join each vertex of R with u by an edge. Evidently the graph B thus obtained is a bipartite graph on the vertex sets C, D such that $|C| = \mathfrak{a}$, $|D| = b$. In this graph $D^* = \{u\}$, thus $|D^*| = 1$. But this graph can be mapped by a bicomplete homomorphism onto the original graph $K_{b,b}$ and hence $\beta_0(B) = b$ (it can be easily proved that it cannot be greater than b).

References

[1] *Harary, F. - Hsu, D. - Miller, Z.:* The bichromaticity of a tree. In: Theory and Applications of Graphs, Proc. Michigan 1976.

[2] *Ore, O.:* Theory of Graphs. Providence 1962.

Author's address: 460 01 Liberec 1, Felberova 2, ČSSR (katedra matematiky VŠST).