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SEMI-AUTOMORPHISMS OF TRANSFORMATION SEMIGROUPS

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1. INTRODUCTION

Ancochea [1] was the first person to consider "semi-automorphisms" of an algebraic structure: he defined a semi-automorphism of a ring R to be an additive bijection $\phi : R \to R$ satisfying

$$(1) \qquad (ab)\,\phi + (ba)\,\phi = a\phi b\phi + b\phi a\phi$$

for all $a, b \in R$, and noted that such mappings arose in connection with certain problems in projective geometry. He proved that if R is a simple algebra that is finite-dimensional over its centre F and char $(F) \neq 2$ then every semi-automorphism of R is either an automorphism or an anti-automorphism. In [10] Jacobson observed that any such algebra R leads in a natural way to a simple Jordan ring R^+, and that ϕ is a semi-automorphism of R if and only if ϕ is an automorphism of R^+; he then deduced Ancochea's result from an investigation of the automorphisms of simple Jordan rings in general.

For any ring R containing an identity 1, Kaplansky [13] defined a semi-automorphism of R to be an additive bijection $\phi : R \to R$ such that

$$(2) \qquad 1\phi = 1 \quad \text{and} \quad (aba)\,\phi = a\phi b\phi a\phi$$

for all $a, b \in R$. Since in any ring $ab + ba = (a + b)^2 - a^2 - b^2$, any semi-automorphism in this latter sense satisfies (1); the two concepts are equivalent when R has no elements of additive order 2, as can be seen using the identity:

$$2aba = 4(a + b)^3 - (a + 2b)^3 - 3a^3 + 4b^3 - 2(a^2b + ba^2)\,.$$

Using his more restrictive definition, Kaplansky showed that every simple algebra with finite dimension over its centre F (without any condition on the characteristic of F) has the property: any semi-automorphism is either an automorphism or an anti-automorphism.

For convenience in this paper we shall say that any algebraic structure with this last-mentioned property "has disjunctive semi-automorphisms", it being understood from the context in what sense "semi-automorphism" is to be taken. The qualification "disjunctive" will also be used in connection with "semi-homomorphisms" and "half-homomorphisms" to be defined below.

Dieudonné ([4] pp 16–17) applied Kaplansky's result in his description of the automorphisms of $GL(2, K)$ for any division ring K with $\mathrm{char}(K) = 2$ and finite dimension over its centre. Hua [9] provided an elementary proof showing that division rings always have disjunctive semi-automorphisms in the sense of (2), and then used this result and his classification of the automorphisms of $SL(2, F)$ for any field F to determine the automorphisms of $GL(2, K)$ for any division ring K ([4] Supplement, pp 96–101).

If R, S are rings (not necessarily with identity) by a semi-homomorphism $\phi : R \to S$ we mean an additive mapping satisfying

$$(3) \qquad a^2\phi = (a\phi)^2 \quad \text{and} \quad (aba)\,\phi = a\phi b\phi a\phi$$

for all $a, b \in R$. Jacobson and Rickart [11] generalised Hua's Theorem to read: every semi-homomorphism from a ring into an integral domain is a homomorphism or an anti-homomorphism. To do this they observed that in any ring

$$abc + cba = (a + c)\,b(a + c) - aba - cbc\,,$$

and consequently for any semi-homomorphism ϕ of rings

$$[(ab)\,\phi - a\phi b\phi]\,.\,[(ab)\,\phi - b\phi a\phi] = 0\,.$$

If R, S are arbitrary rings by a half-homomorphism $\phi : R \to S$ we mean an additive mapping such that for all $a, b \in R$,

$$(4) \qquad (ab)\,\phi \quad \text{equals either} \quad a\phi b\phi \quad \text{or} \quad b\phi a\phi\,.$$

The above-mentioned generalisation now follows from Hua's Lemma ([11] Lemma 1): every ring has disjunctive half-homomorphisms. For another account of these ideas see ([12] pp 110–111, Exercises 7–10).

Jacobson and Rickart also showed in [11] Theorem 8 that if R is a "locally matrix ring" (that is, a ring in which every finite subset can be embedded in a full matrix ring $M_n(S)$ for some ring S and some $n \geqq 2$) then every semi-homomorphism of R is the sum of a homomorphism and an anti-homomorphism (where "sum" is taken in the sense that if $\phi : R \to T$ then $T = T_1 \oplus T_2$ and there exist $\phi_i : R \to T_i$ for $i = 1, 2$ such that $\phi = \phi_1 + \phi_2$). It easily follows from Litoff's Theorem: simple rings with minimal one-sided ideals are division rings or locally matrix rings (compare [11] page 490) and Hua's Theorem on division rings, that simple rings with minimal one-sided ideals have disjunctive semi-automorphisms in the sense of (3). However Jacobson and Rickart extended this result to primitive rings with minimal one-sided ideals (compare [11] Theorem 13) and considered conditions (on the ideal structure of the associated Lie rings) under which every semi-homomorphism from a ring onto a primitive ring with minimal one-sided ideals is a homomorphism or an anti-homomorphism ([11] Theorem 21).

In 1951 Dinkines suggested that just as "ring-theoretic semi-automorphisms" in the sense of (2) had been useful in determining the automorphisms of certain classical groups, a similar concept for groups may also have fruitful applications in the area of, for example, permutation groups. In [5], she proved:

Theorem 1. *If* $|X| \geqq 5$ *then every non-trivial normal subgroup of* Sym(X) *has disjunctive semi-automorphisms,*
and conjectured that "every simple group has disjunctive semi-automorphisms" (for the normal subgroups of infinite symmetric groups see [16]). Subsequently Herstein and Ruchte [7] simplified Dinkines' proof and showed that every non-abelian simple group possessing an element of order 4 has disjunctive semi-automorphisms. We note in passing that $PSL(2, q)$ is simple for $q > 3$ ([16] Theorem 10.8.4) and contains no elements of order 4 whenever q is odd and $x^2 = 2$ has no solution in $GF(q)$ ([23] Exercise IV.8). Moreover, in view of Euler's Criterion: 2 is a quadratic residue modulo an odd prime p if and only if $2^{(p-1)/2} \equiv 1 \pmod p$, and the proof of Theorem 4(5) in [2] pp 295–296, this latter condition holds for an odd prime q precisely when $q \equiv 3 \pmod 8$ or $q \equiv 5 \pmod 8$.

Scott [17] generalised Theorem 1 in a different manner as follows:

Theorem 2. *If X is infinite then every subgroup of* Sym(X) *containing* Alt(X) *has disjunctive semi-automorphisms.*
Since the automorphisms of any infinite permutation group G containing Alt(X) are known to be inner (see [16] Theorem 11.4.6 and [21] Theorem 2), the above result provides a complete description of the semi-automorphisms of G.

We now return to half-homomorphisms in the sense of (4): in [15] Scott proved that cancellative semigroups have disjunctive half-automorphisms and deduced that groups have disjunctive half-homomorphisms; he also provided an example of a non-cancellative semigroup with a proper half-automorphism. Later Sevrin [18, 19] observed that the concept of "half-isomorphism" arose naturally in connection with the problem of deciding when two semigroups have isomorphic subsemigroup-lattices; in [19] he generalised Scott's work to read: any half-isomorphism $\phi : S \to T$ from a cancellative semigroup S into an arbitrary semigroup T is either an isomorphism or an anti-isomorphism.

2. SEMI-AUTOMORPHISMS OF SEMIGROUPS

Notation will be that of [3], [16] and [20]. In particular a transformation semigroup S is any subsemigroup of $\mathscr{P}_X$ and $K(S)$ denotes all the constants of S with $\square$ adjoined. Moreover, we say S covers X when for each $x \in X$ the semigroup S contains a constant idempotent (denoted by A_x for some $A \subseteq X$) with range $\{x\}$. Any bijection $\phi : S \to S$ such that $(aba)\phi = a\phi b\phi a\phi$ for all $a, b \in S$ will be called a *semi-automorphism of S*. Symons [22] has shown that if S is a total transformation semigroup that covers X then every semi-automorphism of S is an automorphism (and hence "inner" by [20] Theorem 1). We shall show that a similar conclusion holds for any 2-transitive (partial) transformation semigroup with a stronger covering property. Our first result in this direction is comparable with [14] Lemma 2.4.

Lemma 1. *If S is a transitive transformation semigroup covering X and $\lambda \in S$ then λ is a constant if and only if $\lambda \neq \square$ and $\lambda\alpha\lambda$ equals λ or $\square$ for each $\alpha \in S$.*

Proof. If $\lambda \neq \square$ choose $x \in \operatorname{dom} \lambda$, $y \in \operatorname{ran} \lambda$, $\alpha \in S$ with $y\alpha = x$ and $A_x \in S$ with $x \in A$. Then $\alpha A_x = \beta$ (say) maps y to x and $\lambda\beta\lambda \neq \square$. Hence $\lambda = \lambda\beta\lambda$ which is a constant. The converse is obvious.

If $\phi : S \to S$ is a semi-automorphism of an arbitrary semigroup $S = S^0$ and if $a\phi = 0$ then $0\phi = (0a0)\,\phi = 0\phi \,.\, 0 \,.\, 0\phi = 0$ implies that $a = 0$; we shall use this fact in what follows.

Lemma 2. *If S is a transitive transformation semigroup covering X and ϕ is a semi-automorphism of S then ϕ maps $K(S)$ onto $K(S)$.*

Proof. Let $x \in X$. Choose $A_x \in S$ and put $A_x\phi = \lambda$ (in which case $\lambda \neq \square$). If $\alpha \in S$ and $\alpha = \beta\phi$ then $\lambda\alpha\lambda = (A_x \,.\, \beta \,.\, A_x)\,\phi$ where $A_x \,.\, \beta \,.\, A_x$ equals A_x or $\square$. Hence for all $\alpha \in S$, $\lambda\alpha\lambda$ equals λ or $\square$ in which case λ must be a constant by Lemma 1. To complete the proof we simply note that if ϕ is a semi-automorphism of S then ϕ^{-1} is also.

Symons' result can easily be deduced from the above Lemma. However to consider semi-automorphisms of partial transformation semigroups, we need the following definition: S is 2-transitive on X if for all distinct x, y and distinct a, b in X there exists $\alpha \in S$ such that $x\alpha = a$ and $y\alpha = b$.

Lemma 3. *If S is 2-transitive and contains all the total constants and $\lambda \in S \setminus \square$ then λ is a total constant if and only if $\lambda = \lambda^3$ and for all $\alpha \in S$, $\alpha\lambda\alpha \neq \square$ implies $\lambda\alpha\lambda = \lambda$.*

Proof. If $a \in \operatorname{dom} \lambda$ then $X_a \,.\, \lambda \,.\, X_a \neq \square$ implies $\lambda = \lambda \,.\, X_a \,.\, \lambda$, a constant C_z (say). Let $x \in X \setminus z$ and choose $\alpha \in S$ satisfying $z\alpha = x$, $x\alpha = z$. Then $\alpha\lambda\alpha \neq \square$ (since $\lambda = \lambda^3$ implies $z \in C$) and so $\lambda = \lambda\alpha\lambda$; it follows that $x \in C$ and so $C = X$ as required. The converse is obvious.

We shall say that S extremally covers X if S contains all the total constants X_a, $a \in X$, and all the injective constants a_b, $a, b \in X$.

Theorem 3. *If S is 2-transitive and extremally covers X then every semi-automorphism of S is an inner automorphism.*

Proof. By Lemmas 2 and 3, for each semi-automorphism ϕ of S we can define $g \in \operatorname{Sym}(X)$ such that

$$ag = b \quad \text{if and only if} \quad X_a\phi = X_b\,.$$

We now assert that for all $a, b \in X$, $a_b\phi = ag_{bg}$. For, suppose $A_x\phi = c_d$ and let $z \in A$. Then

$$X_z\phi = (X_z \,.\, A_x \,.\, X_z)\,\phi = X_{zg} \,.\, c_d \,.\, X_{zg}\,,$$

so that $zg = c$. Thus $A = \{z\}$ and since ϕ^{-1} is also a semi-automorphism we have $a_b\phi = ag_d$ for some $d \in X$. Hence

$$a_b\phi = a_b\phi \,.\, b_a\phi \,.\, a_b\phi = ag_d \,.\, bg_e \,.\, ag_d$$

for some $e \in X$ and we have $d = bg$ as asserted.

From this it follows that if $C_a \in S$ then $C_a\phi = D_{ag}$ for some $D \subseteq X$. For, if $c \in C$ then

$$a_c\phi = (a_c \,.\, C_a \,.\, a_c)\,\phi = ag_{cg} \,.\, D_y \,.\, ag_{cg}$$

for some $y \in X$ and $D \subseteq X$; that is, $y = ag$ as required.

Now let $\alpha \in S$ and $a \in \operatorname{dom} \alpha$. From the preceding remarks we have

$$\alpha\phi \,.\, X_{ag.\alpha\phi} = (\alpha \,.\, X_a \,.\, \alpha)\,\phi = C_{a\alpha}\phi = C_{a\alpha g}$$

for some $C \subseteq X$, and so $(\operatorname{dom} \alpha)\, g \subseteq \operatorname{dom}(\alpha\phi)$. A converse argument using ϕ^{-1} establishes equality, so that $\alpha\phi = g^{-1}\alpha g$ for all $\alpha \in S$.

To obtain a result for transformation semigroups that is closer to Dinkines' original work on permutation groups we now restrict our attention to $\mathscr{I}_X$ and note that the mapping $\theta : \mathscr{I}_X \to \mathscr{I}_X$, $\alpha \to \alpha^{-1}$, is an anti-automorphism of $\mathscr{I}_X$. Moreover, if ϕ is a semi-automorphism of any inverse semigroup S then $a^{-1}\phi = (a\phi)^{-1}$ for all $a \in S$.

Theorem 4. *If S is a 2-transitive inverse subsemigroup of $\mathscr{I}_X$ covering X then a semi-automorphism of S is either an automorphism (in which case it is inner) or an anti-automorphism (in which case it is the composition of θ and an inner automorphism).*

Proof. Let ϕ be a semi-automorphism of S. By Lemma 2 we can define $g \in \operatorname{Sym}(X)$ by

$$xg = y \quad \text{if and only if} \quad x_x\phi = y_y\,.$$

Suppose $x \neq y$ and $x_y\phi = a_b$. If $a \neq xg$ and $b \neq xg$ choose $\lambda \in S$ with $(xg)\,\lambda = a$ and $b\lambda = xg$. If $\alpha\phi = \lambda$ then, since λ is 1-1, we have

$$x_x\phi = xg_{xg} = \lambda \,.\, a_b \,.\, \lambda = (\alpha \,.\, x_y \,.\, \alpha)\,\phi\,,$$

and so $y\alpha = x$, $\alpha \,.\, x_y \,.\, \alpha = y_x = x_x$, a contradiction. Hence either $a = xg$ or $b = xg$; since $x \neq y$ and $y_x\phi = b_a$ we therefore conclude that $x_y\phi$ equals xg_{yg} or yg_{xg}: in the first case $\alpha\phi = g^{-1}\alpha g$ for all $\alpha \in S$ and in the second $\alpha\phi = g^{-1}(\alpha\theta)\, g$ for all $\alpha \in S$.

For, suppose $x_y\phi = xg_{yg}$ for some $x \neq y$ in X and let $z \in X \setminus \{x, y\}$. If $x_z\phi = zg_{xg}$ choose $\lambda \in S$ so that $(xg)\,\lambda = xg$, $(yg)\,\lambda = zg$ and let $\alpha\phi = \lambda$. Then

$$y_x\phi = yg_{xg} = \lambda \,.\, zg_{xg} \,.\, \lambda = (\alpha \,.\, x_z \,.\, \alpha)\,\phi$$

and so $z\alpha = x = y\alpha$; this implies $y = z$, contradicting the choice of x, y, z. Hence $x_z\phi = xg_{zg}$ for all $z \in X$. Now suppose distinct $a, b \in X \setminus x$ and $a_b\phi = bg_{ag}$. Choose $\mu \in S$ with $(xg)\mu = bg$, $(ag)\mu = ag$ and let $\beta\phi = \mu$. Then

$$x_a\phi = xg_{ag} = \mu \,.\, bg_{ag} \,.\, \mu = (\beta \,.\, a_\beta \,.\, \beta)\,\phi$$

and so $b\beta = a = x\beta$, a contradiction. We have therefore shown that either $a_b\phi = ag_{bg}$ for all $a, b \in X$ or $a_b\phi = bg_{ag}$ for all $a, b \in X$. In the first case let $\alpha \in S$, $b \in \operatorname{dom}\alpha$, $a \in \operatorname{ran}\alpha$ and $x\alpha = a$. Then

$$\alpha\phi \,.\, ag_{bg\alpha\phi} = (\alpha \,.\, a_b \,.\, \alpha)\,\phi = x_{b\alpha}\phi = xg_{b\alpha g}$$

and the argument is completed as in the proof of Theorem 3. For the second case put $\alpha\psi = g \,.\, \alpha\phi \,.\, g^{-1}$ for all $\alpha \in S$: we have to show that $\psi = \theta$. To do this note that with the above notation

$$\alpha\psi \,.\, b_{a\alpha\psi} = (\alpha \,.\, a_\beta \,.\, \alpha)\,\psi = x_{b\alpha}\psi = b\alpha_x$$

from which we obtain $\operatorname{dom}\alpha^{-1} = \operatorname{dom}(\alpha\psi)$ and the result follows.

Symons' result and the foregoing Theorems 3 and 4 can be readily applied to the semigroups $\mathscr{T}_X$ and $\mathscr{E}_X$ (the semigroup generated by all the idempotents of $\mathscr{T}_X$: see [8]), $\mathscr{P}_X$ and $\mathscr{F}_X$ (the semigroup generated by all the idempotents of $\mathscr{P}_X$: see [6]) or to $\mathscr{I}_X$.

3. HALF-AUTOMORPHISMS OF SEMIGROUPS

If S is a semigroup, $\phi : S \to S$ is a half-automorphism of S if it is bijective and for all $a, b \in S$, $(ab)\,\phi$ equals either $a\phi b\phi$ or $b\phi a\phi$. Symons [22] has shown that if S is a 2-transitive total transformation semigroup covering X then every half-automorphism of S is an automorphism (and hence inner); he also gives an example of a total transformation semigroup admitting a half-automorphism which is neither an automorphism nor an anti-automorphism, but which nonetheless contains all the total constants (and hence is transitive). We conjecture that Symons' result can be extended to any 2-transitive (partial) transformation semigroup extremally covering X. Before stating a result providing some support for this claim we note that if $S = S^0$ and ϕ is a half-automorphism of S then $0\phi = 0$. For, if $\alpha\phi = 0$ then $(0 \,.\, a)\,\phi$ equals $0\phi \,.\, 0$ or $0 \,.\, 0\phi$, and this implies $0\phi = 0$ and moreover $a = 0$ as required.

Lemma 4. *If S covers X, $\phi : S \to S$ is a half-automorphism and α is an idempotent constant in S then $\alpha\phi$ is also.*

Proof. Let $A_x \in S$ with $x \in A$ and $A_x\phi = \lambda$. Then $\lambda^2 = \lambda \neq \square$ and we can choose $y \in \operatorname{ran}\lambda$, $B_y \in S$ with $y \in B$, and put $\alpha\phi = B_y$ (in which case $\alpha^2 = \alpha$). Now $(A_x\alpha)\,\phi$ equals $\lambda \,.\, B_y$ or $B_y \,.\, \lambda$, both of which are non-zero: hence $x \in \operatorname{dom}\alpha$. If $(A_x\alpha)\,\phi = \lambda B_y = C_y$ (say) then $y \in C_y$ and $x\alpha \in A$; thus, $\lambda = (A_{x\alpha}A_x)\,\phi$ equals $C_y\lambda$ or λC_y, both of which are constant. On the other hand, if $(A_x\alpha)\,\phi = B_y\lambda = B_y$ then $\alpha = A_z$ for some $z \in A$ and the result follows as before.

References

[1] *G. Ancochea:* On semi-automorphisms of division algebras, Ann. of Math., *48* (1947) 147—153.

[2] *L. Childs:* A Concrete Introduction to Higher Algebra, Springer-Verlag, 1979.

[3] *A. H. Clifford* and *G. B. Preston:* The Algebraic Theory of Semigroups, Math. Surveys, No. 7, Amer. Math. Soc., Providence, RI, Volume *1* (1961), Volume *2* (1967).

[4] *J. Dieudonné:* On the automorphisms of the classical groups, Mem. Amer. Math. Soc., No. *2*, 1951.

[5] *F. Dinkines:* Semi-automorphisms of symmetric and alternating groups, Proc. Amer. Math. Soc., *2* (1951) 478—486.

[6] *A. E. Evseev* and *N. E. Podran:* Semigroups of transformations generated by idempotents with given projection characteristics, Izv. Vyss. Ucebn. Zaved. Mat., *12* (103) 1970, 30—36.

[7] *I. N. Herstein* and *M. F. Ruchte:* Semi-automorphisms of groups, Proc. Amer. Math. Soc., *9* (1958) 145—150.

[8] *J. M. Howie:* The subsemigroup generated by the idempotents of a full transformation semigroup, J. London Math. Soc., *41* (1966) 707—716.

[9] *Loo-Keng Hua:* On the automorphisms of a sfield, Proc. Nat. Acad. Sci. U.S.A., *35* (1949) 386—389.

[10] *N. Jacobson:* Isomorphisms of Jordan rings, Amer. J. Math., *70* (1948) 317—326.

[11] *N. Jacobson* and *C. E. Rickart:* Jordan homomorphisms of rings, Trans. Amer. Math. Soc., *69* (1950) 479—502.

[12] *N. Jacobson:* Basic Algebra, I, W. H. Freeman and Co., 1974.

[13] *I. Kaplansky:* Semi-automorphisms of rings, Duke Math. J., *14* (1947) 521—525.

[14] *K. D. Magill:* Semigroup structures for families of functions, I, J. Austral. Math. Soc., *7* (1967) 81—94.

[15] *W. R. Scott:* Half-homomorphisms of groups, Proc. Amer. Math. Soc., *8* (1957) 1141—1144.

[16] *W. R. Scott:* Group Theory, Prentice-Hall, 1964.

[17] *W. R. Scott:* Semi-isomorphisms of certain infinite permutation groups, Proc. Amer. Math. Soc., *21* (1969) 711—713.

[18] *L. N. Sevrin:* Semi-isomorphisms and lattice isomorphisms of semigroups with a cancellation law, Dokl. Akad. Nauk SSSR, *171* (1966) 296—298; translated as Sov. Math. Dokl., *7* (1966) 1491—1493.

[19] *L. N. Sevrin:* Semi-isomorphisms of semigroups with cancellation law, Izv. Akad. Nauk SSSR, Ser. Mat. *31* (1967) 957—964.

[20] *R. P. Sullivan:* Automorphisms of transformation semigroups, J. Austral. Math. Soc., *20* (Series A) 1975, 77—84.

[21] *R. P. Sullivan:* Automorphisms of injective transformation semigroups (to appear).

[22] *J. S. V. Symons:* Automorphisms of transformation semigroups, Ph. D. Thesis, University of Western Australia, 1973.

[23] *M. Weinstein:* Examples of Groups, Polygonal Pub. House, 1977.

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