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ON DETERMINATION OF A CYCLIC ORDER

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In [5] it is shown that there exists a close relation between cyclic orders and orders on a set $G$. The aim of this paper is the study of cyclic orders from this point of view. We show that any cyclic order is in a certain sense generated by a system of orders. Further, the so-called cocyclic order is introduced and properties of this relation are studied.

1. ORDERS AND CYCLIC ORDERS

1.1. Remark. By an ordered set we mean a pair $(G, <)$ where $G$ is a set and $<$ is an order on $G$, i.e., an irreflexive and transitive binary relation on $G$. If $(G, <)$ is an ordered set, then there exists the least (with respect to the set inclusion) subset $H$ of $G$ such that $< \subseteq H^2$. If $<$ is a linear order on this set $H$, then we shall call the order $<$ a linear order in $G$.

1.2. Definition. Let $G$ be a set, $C$ a ternary relation on $G$. $C$ is called a cyclic order on $G$, iff it is:

(i) asymmetric, i.e. $(x, y, z) \in C \Rightarrow (z, y, x) \notin C$,
(ii) transitive, i.e. $(x, y, z) \in C, (x, z, u) \in C \Rightarrow (x, y, u) \in C$,
(iii) cyclic, i.e. $(x, y, z) \in C \Rightarrow (y, z, x) \in C$.

If $G$ is a set and $C$ a cyclic order on $G$, then the pair $(G, C)$ is called a cyclically ordered set. If, moreover, card $G \geq 3$ and $C$ is

(iv) complete, i.e. $x, y, z \in G, x \neq y \neq z \neq x \Rightarrow (x, y, z) \in C$ or $(z, y, x) \in C$,
then $C$ is called a linear cyclic order on $G$ and $(G, C)$ is called a linearly cyclically ordered set or a cycle. If $C = \emptyset$, then $(G, C)$ is called a discrete cyclically ordered set.

1.3. Lemma. Let $(G, <)$ be an ordered set. For any $x, y, z \in G$ put $(x, y, z) \in C_<$ iff either $x < y < z$ or $y < z < x$ or $z < x < y$. Then $C_<$ is a cyclic order on $G$.

Proof. Trivial; see also [5], 3.5.
1.4. Lemma. Let \((G, C)\) be a cyclically ordered set, \(x \in G\). For any \(y, z \in G\) put 
\[ y <_{C,x} z \iff \text{either } (x, y, z) \in C \text{ or } x = y \neq z. \]
Then \(<_{C,x}\) is an order on \(G\) and \(x\) is the least element of \((G, <_{C,x})\).

Proof. Trivial; see also [5], 3.1.

1.5. Lemma. Let \((G, \prec)\) be an ordered set with the least element \(x\). Then there exists a cyclic order \(C\) on \(G\) such that \(\prec = <_{C,x}\).

Proof. Put \(C = C_{\prec}\). By 1.3, \(C\) is a cyclic order on \(G\); it is not difficult to prove 
\[ < = <_{C,x} \text{ (see also [5], 3.8).} \]
Now we can put an analogous question: Let \((G, C)\) be a cyclically ordered set. Does there exist an order \(<\) on \(G\) such that \(C = C_{\prec}\)? The following lemma shows that the answer is negative in general.

1.6. Lemma. Let \((G, C)\) be a cyclically ordered set. If there exists an order \(<\) on \(G\) such that \(C = C_{\prec}\), then there exists a linear extension of \(C\) on \(G\), i.e. such a linear cyclic order \(D\) on \(G\) that \(C \subseteq D\).

Proof. According to Szpirojan’s theorem ([7]) there exists a linear extension \(<\) of the order \(<\) on \(G\). Thus \(< \equiv <\) and hence \(C_{\prec} \subseteq C_{\prec} \equiv C_{\prec} \subseteq C_{\prec}\). But \(C_{\prec}\) is evidently a linear cyclic order on \(G\).

As there exist cyclic orders that have no linear extension ([3]), 1.6 generally implies the negative answer to the above question. Nevertheless, we shall show that any cyclic order is a union of cyclic orders, each of which is generated by an order according to 1.3.

1.7. Definition. Let \(G\) be a set, \((<_{i})_{i \in I}\) an indexed family of orders on \(G\). We call this family harmonized iff the following conditions hold:

1. If \(i \in I\) and \(x, y, z \in G\) are such elements that \(x <_{i} y <_{i} z\), then either \(z <_{j} y, y <_{j} x, x <_{j} z\) or \(x <_{j} z, z <_{j} y\) for any \(j \in I\).

2. If \(i, j \in I\) and \(x, y, z, u \in G\) are such elements that \((x, y, z) \in C_{<_{i}}, (x, z, u) \in C_{<_{j}}\), then there exists \(k \in I\) such that \((x, y, u) \in C_{<_{k}}\).

1.8. Theorem. Let \(G\) be a set, \((<_{i})_{i \in I}\) a family of orders on \(G\). Then the following statements are equivalent:

(A) The family \((<_{i})_{i \in I}\) is harmonized.

(B) The ternary relation \(C = \bigcup_{i \in I} C_{<_{i}}\) is a cyclic order on \(G\).

Proof. 1. Let (A) hold. If \((x, y, z) \in C\), then there exists \(i \in I\) such that \((x, y, z) \in C_{<_{i}}\), i.e. either \(x <_{i} y <_{i} z\) or \(y <_{i} z <_{i} x\) or \(z <_{i} x <_{i} y\). Suppose \((z, y, x) \in C\); then there exists \(j \in I\) such that \((z, y, x) \in C_{<_{j}}\), i.e. either \(z <_{j} y <_{j} x\) or \(y <_{j} x <_{j} z\) or \(x <_{j} z <_{j} y\). By a simple calculation we find that this contradicts (1) of 1.7 in all cases. The relation \(C\) is thus asymmetric. Trivially, (2) of 1.7 implies that the relation
C is transitive. Finally, as all relations $C_{<i}$ (i.e., $i \in I$) are cyclic, the union $C = \bigcup_{i \in I} C_{<i}$ is cyclic as well. Hence $C$ is a cyclic order on $G$ and (B) holds.

2. Let (B) hold. If $i \in I$, $x, y, z \in G$ are such elements that $x <_i y <_i z$, then $(x, y, z) \in C_{<i} \subseteq C$ so that $(z, y, x) \in C$. This means $(z, y, x) \in C_{<j}$ for any $j \in I$, i.e., neither $z <_j y <_j x$ nor $y <_j x <_j z$ nor $x <_j z <_j y$ holds and this implies (1) of 1.7. Further, the transitivity of $C$ implies (2) of 1.7. Thus, the family $(<_i)_{i \in I}$ is harmonized and (A) holds.

1.9. Theorem. Let $(G, C)$ be a cyclically ordered set. Then there exists a family $(<_i)_{i \in I}$ of orders on $G$ such that $C = \bigcup_{i \in I} C_{<i}$.

Proof. Let $\mathcal{F}$ be the set of all ordered triples $(x, y, z) \in G^3$ such that $(x, y, z) \in C$. For any $\tau = (x, y, z) \in \mathcal{F}$ define an order $<_\tau$ on $\{x, y, z\}$ by $x <_\tau y <_\tau z$. Then $(<_\tau)_{\tau \in \mathcal{F}}$ is a family of orders on $G$ and clearly $C = \bigcup_{\tau \in \mathcal{F}} C_{<_\tau}$ holds.

Let us note that any order $<_\tau$ in the proof of 1.9 is a linear order in $G$. Thus, a stronger result holds:

1.10. Corollary. Let $(G, C)$ be a cyclically ordered set. Then there exists a family $(<_\tau)_{\tau \in \mathcal{F}}$ of linear orders in $G$ such that $C = \bigcup_{\tau \in \mathcal{F}} C_{<_\tau}$.

From 1.8 it follows that the family $(<_\tau)_{\tau \in \mathcal{F}}$ in the proof of 1.9 is harmonized; naturally it is simple to prove it directly. But we prove also

1.11. Theorem. Let $(G, C)$ be a cyclically ordered set. Then $C = \bigcup_{x \in G} C_{<c,x}$.

Proof. It is not difficult to prove $C_{<c,x} \subseteq C$ for any $x \in G$ (see also [5], 3.9). Thus we have $\bigcup_{x \in G} C_{<c,x} \subseteq C$. On the other hand, if $(x, y, z) \in C$, then $x <_{c,x} y <_{c,x} z$, which implies $(x, y, z) \in C_{<c,x}$. This yields $C \subseteq \bigcup_{x \in G} C_{<c,x}$ and hence $C = \bigcup_{x \in G} C_{<c,x}$.

1.12. Corollary. Let $(G, C)$ be a cyclically ordered set. Then the family $(<_{c,x})_{x \in G}$ of orders on $G$ is harmonized.

2. WIDTH OF A CYCLICALLY ORDERED SET

2.1. Definition. Let $(G, C)$ be a cyclically ordered set. We put $w(G, C) = \min \{\text{card } I; \text{there exists a harmonized family } (<_i)_{i \in I} \text{ of orders on } G \text{ such that } C = \bigcup_{i \in I} C_{<_i}\}$, $W(G, C) = \min \{\text{card } I; \text{there exists a harmonized family } (<_i)_{i \in I} \text{ of linear orders in } G \text{ such that } C = \bigcup_{i \in I} C_{<_i}\}$. The number $w(G, C)$ will be called the width, the number $W(G, C)$ the strong width of $(G, C)$. 557
If $T$ is a ternary relation on a set $G$, then we denote by $T^c$ the cyclic hull of $T$, i.e. $T^c = \{(x, y, z) \in G^3 : \text{there exists an even permutation } (u, v, w) \text{ of the sequence } (x, y, z) \text{ such that } (u, v, w) \in T\}$. 

2.2 Example. Let $G = \{x, y, z, u, v\}$, $T = \{(x, y, z), (x, y, u), (x, y, v), (z, u, v)\}$, $C = T^c$ (Fig. 1). It is easy to see that $C$ is a cyclic order on $G$; we shall show $w(G, C) = 2$, $W(G, C) = 4$.

First, we show that $w(G, C) > 1$. Suppose $w(G, C) = 1$, i.e. there exists an order $<$ on $G$ such that $C = C_\prec$. Then $(x, y, z) \in C_\prec$, thus either $x < y < z$ or $y < z < x$ or $z < x < y$, and simultaneously $(x, y, u) \in C_\prec$, $(x, y, v) \in C_\prec$, $(z, u, v) \in C_\prec$.

Case 1. Let $x < y < z$. Then $y < u < x$ is impossible. If $u < x < y$, then $u < x < z$, thus $(u, x, z) \in C_\prec = C$, a contradiction. Hence we have $x < y < u$. For the same reason $x < y < v$ holds. If $z < u < v$, then $y < z < u$ and $(y, z, u) \in C_\prec = C$; if $u < v < z$, then $y < u < v$ and $(y, u, v) \in C$; if $v < z < u$, then $y < v < z$ and $(y, v, z) \in C$. Thus the case $x < y < z$ is impossible.

Case 2. Let $y < z < x$. Then $x < y < u$, $u < x < y$ are impossible, hence $y < u < x$. Analogously $y < v < x$ holds. If $z < u < v$, then $y < z < u$ and $(y, z, u) \in C$; if $u < v < z$, then $y < u < z$ and $(y, u, z) \in C$; if $v < z < u$, then $y < z < u$ and $(y, z, u) \in C$. Thus also the case $y < z < x$ is impossible.

Case 3. Let $z < x < y$. Analogously as in Case 1, we find that $u < x < y$, $v < x < y$ hold and any of the possibilities $z < u < v$, $u < v < z$, $v < z < u$ leads to a contradiction. Thus we have shown $w(G, C) > 1$. Now put $<_1 = \{(x, y), (x, z), (y, z), (x, u), (y, u), (x, v), (y, v)\}$, $<_2 = \{(z, u), (z, v), (u, v)\}$ (Fig. 2). We easily see that $C_{<_1} \cup C_{<_2} = C$. Thus $w(G, C) = 2$. 

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Further, put $G_1 = \{x, y, z\}$, $G_2 = \{x, y, u\}$, $G_3 = \{x, y, v\}$, $G_4 = \{z, u, v\}$ and let us define a linear order $<_i$ on $G_i$ ($i = 1, 2, 3, 4$) as follows: $x <_1 y <_1 z, x <_2 y <_2 z$. Each $<_i$ is a linear order in $G$ and clearly $\bigcup_{i=1}^4 C_{<_i}$. This implies $W(G, C) \leq 4$. On the other hand, let $(<_i)_{i \in I}$ be a family of linear orders in $G$ such that $C = \bigcup_{i \in I} C_{<_i}$ and let $i \in I$ be such an element that $(x, y, z) \in C_{<_i}$. Then $<_i$ is a linear order on $H \cong G_1$; if $H \not\cong G_1$, then either $u \in H$ or $v \in H$. In the first case we have either $(y, z, u) \in C_{<_i}$ or $(u, z, y) \in C_{<_i}$, which is a contradiction, for $(y, z, u) \in C$, $(u, z, y) \in C$; in the second, either $(y, z, v) \in C_{<_i}$ or $(v, z, y) \in C_{<_i}$, a contradiction. Thus $H = G_1$. For the same reason there exist $j \in I, j \neq i, k \in I, i \neq k \neq j, l \in I, l \in \{i, j, k\}$ such that $<_j$ is a linear order on $G_2$, $<_k$ a linear order on $G_3$, $<_l$ a linear order on $G_4$. Thus card $I \geq 4$, $W(G, C) \geq 4$ and we have $W(G, C) = 4$.

The definition directly yields

2.3. Lemma. Let $(G, C)$ be a cyclically ordered set. Then
(1) $w(G, C) \leq W(G, C)$,
(2) $w(G, C) = 1$ iff there exists an order $<$ on $G$ such that $C = C_<$.

In [5], the following notion was introduced (3.12): A cyclically ordered set $(G, C)$ is called $x$-stable (where $x \in G$) iff the following condition is satisfied: $y, z \in G - \{x\}$, $(u, y, z) \in C$ for some $u \in G \Rightarrow (x, y, z) \in C$ or $(z, y, x) \in C$. Further, it is proved that (3.15) if $(G, C)$ is a cyclically ordered set which is $x$-stable for some $x \in G$, then $C = C_{<_C x}$. As a consequence, we obtain

2.4. Corollary. Let $(G, C)$ be a cyclically ordered set. If there exists $x \in G$ such that $(G, C)$ is $x$-stable, then $w(G, C) = 1$.  

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Let us recall the definition of the direct sum of cyclically ordered sets ([5], 2.7): Let \((G_i, C_i)_{i \in I}\) be a family of cyclically ordered sets and let the sets \(G_i\) be pairwise disjoint. The direct sum of sets \((G_i, C_i)\) \((i \in I)\) is the cyclically ordered set \((G, C)\) where \(G = \bigcup_{i \in I} G_i\), \(C = \bigcup_{i \in I} C_i\); we write \((G, C) = \sum_{i \in I} (G_i, C_i)\). If \(I = \{1, \ldots, n\}\), we write \(\sum_{i \in I} (G_i, C_i) = (G_1, C_1) + \cdots + (G_n, C_n)\). Now, let \((G, C)\) be a cyclically ordered set with \(W(G, C) = 1\). Then there exists a linear order \(<\) on a subset \(G_1 \subseteq G\) such that \(C = C_{<}\). If \(G_1 \leq 2\), then \(C_{<} = \emptyset\) so that \((G, C)\) is discrete. If \(G_1 = G\) and \(\text{card } G \geq 3\), then \(C_{<}\) is linear, so that \((G, C)\) is a cycle. In the other cases, put \(G_2 = G - G_1\), \(C_1 = C = C_{<}\), \(C_2 = \emptyset\); then clearly \((G, C) = (G_1, C_1) + (G_2, C_2)\). On the other hand, if \((G, C) = (G_1, C_1) + (G_2, C_2)\) where \((G_1, C_1)\) is a cycle and \((G_2, C_2)\) is discrete, then \(C = C_{<_{C_x}}\) for any \(x \in G_1\) and \(<_{C_x}\) is a linear order in \(G\). Thus, we have

2.5. Lemma. Let \((G, C)\) be a cyclically ordered set. Then \(W(G, C) = 1\) iff \((G, C)\) is either a cycle or a discrete cyclically ordered set or \((G, C) = (G_1, C_1) + (G_2, C_2)\) where \((G_1, C_1)\) is a cycle and \((G_2, C_2)\) is discrete.

2.6. Theorem. Let \((G, C)\) be a cyclically ordered set. Then \(w(G, C) \leq \text{card } G\).

Proof follows from 1.11.

If \((G, C)\) is a cyclically ordered set and \(H \subseteq G\) is a subset such that \(D = C \cap H^3\) is a linear cyclic order on \(H\), then \((H, D)\) is called a cycle in \((G, C)\).

2.7. Theorem. Let \((G, C)\) be a cyclically ordered set which is not discrete. Then \(W(G, C) = \min \{\text{card } I; \text{there exists a family } (G_i, C_i)_{i \in I} \text{ of cycles in } (G, C)\} \text{ such that } C = \bigcup_{i \in I} C_i\).

Proof. Put \(\min \{\text{card } I; \text{there exists a family } (G_i, C_i)_{i \in I} \text{ of cycles in } (G, C)\} = m\). Let \(<_j\>_{j \in J}\) be a harmonized family of linear orders in \(G\) such that \(C = \bigcup_{j \in J} C_{<_j}\) and card \(J = W(G, C)\). Each \(<_j\>\) is a linear order on a certain (maximal) subset \(G_j \subseteq G\) and we may assume card \(G_j \geq 3\) (otherwise \(C_{<_j} = \emptyset\) and \(<_j\) can be omitted). Thus \((G_j, C_{<_j})\) is a cycle in \((G, C)\) and we obtain \(m \leq W(G, C)\). On the other hand, let \((G_i, C_i)_{i \in I}\) be a family of cycles in \((G, C)\) such that \(C = \bigcup_{i \in I} C_i\) and card \(I = m\). By 2.5, \(W(G_i, C_i) = 1\) for each \(i \in I\), i.e. there exists a linear order \(<_i\) in \(G_i\) such that \(C_i = C_{<_i}\). Then each \(<_i\) is a linear order in \(G\) and \(C = \bigcup_{i \in I} C_{<_i}\), which implies \(W(G, C) \leq m\).

2.8. Corollary. Let \((G, <)\) be an ordered set and \((<_i)_{i \in I}\) a family of all maximal linear orders in \(G\) that are included in \(<\). Then \(W(G, C_{<}) \leq \text{card } I\).
Proof. Clearly, \( \bigcup_{i \in I} \prec_i = \prec \) which implies \( \bigcup_{i \in I} C_{\prec_i} \subseteq C_\prec \). On the other hand, if \((x, y, z) \in C_\prec\), then either \(x < y < z\) or \(y < z < x\) or \(z < x < y\). Then there exists a maximal chain \((G_i, \prec_i)\) in \((G, \prec)\) containing \(\{x, y, z\}\) and hence \((x, y, z) \in C_{\prec_i}\). Thus \(C_\prec = \bigcup_{i \in I} C_{\prec_i}\), and the assertion follows from 2.7.

2.9. Lemma. Let \((G, C)\) be a cyclically ordered set, let \(H \subseteq G\) and \(D = C \cap H^3\). Then \(w(H, D) \leq w(G, C)\), \(W(H, D) \leq W(G, C)\).

Proof. Let \((\prec_i)_{i \in I}\) be a harmonized family of orders on \(G\) such that \(C = \bigcup_{i \in I} C_{\prec_i}\) and \(\text{card} I = w(G, C)\). Put \(\prec = \prec_i \cap H^2\); then \(\prec_i\) is an order on \(H\) and it is easy to prove \(D = \bigcup_{i \in I} C_{\prec_i}\). Thus \(w(H, D) \leq \text{card} I = w(G, C)\). If \(\prec_i\) is a liner order in \(G\), then \(\prec_i\) is a linear order in \(H\) so that also \(W(H, D) \leq W(G, C)\).

2.10. Theorem. Let \((G_i, C_i)_{i \in I}\) be a family of cyclically ordered sets and let the sets \(G_i\) be pairwise disjoint. Let \((G, C) = \bigcup_{i \in I} (G_i, C_i)\). Then \(w(G, C) = \sup \{w(G_i, C_i); i \in I\}\) and \(W(G, C) = \sum_{i \in I} W(G_i, C_i)\). If, moreover, no \((G_i, C_i)\) is discrete, then \(W(G, C) = \sum_{i \in I} W(G_i, C_i)\).

Proof. As \(G_i \subseteq G, C_i = C_i \cap G_i^3\) for any \(i \in I\), 2.9. implies \(w(G_i, C_i) \leq w(G, C)\) for any \(i \in I\) and thus \(\sup \{w(G_i, C_i); i \in I\} \leq w(G, C)\). Put \(\sup \{w(G_i, C_i); i \in I\} = m\) and let \(J\) be a set with \(\text{card} J = m\). For any \(i \in I\) find a family \((\prec_{i,j})_{j \in J}\) of orders on \(G_i\) such that \(C_i = \bigcup_{j \in J} C_{\prec_{i,j}}\) and for a given \(j \in J\) put \(\prec_j = \bigcup_{i \in I} \prec_{i,j}\). Then \(\prec_j\) is an order on \(G\) (in fact, \(\prec_j\) is the cardinal sum of orders \(\prec_{i,j}; i \in I\)). We show that \(C = \bigcup_{j \in J} C_{\prec_j}\). Let \((x, y, z) \in C\). Then there exists (just one) \(i \in I\) such that \((x, y, z) \in G_i\) and \((x, y, z) \in C_i\). This implies the existence of \(j \in J\) such that \((x, y, z) \in C_{\prec_{i,j}}\). As \(\prec_{i,j} \subseteq \prec_j\), we have \((x, y, z) \in C_{\prec_j} \subseteq \bigcup_{j \in J} C_{\prec_j}\). We have proved \(C \subseteq \bigcup_{j \in J} C_{\prec_j}\). Let \((x, y, z) \in \bigcup_{j \in J} C_{\prec_j}\). Then there exists \(j \in J\) such that \((x, y, z) \in C_{\prec_j}\). By definition of the order \(\prec_j\) there exists (just one) \(i \in I\) such that \((x, y, z) \in C_{\prec_{i,j}}\). Thus \((x, y, z) \in C_i\) and \((x, y, z) \in C\). Hence \(C = \bigcup_{j \in J} C_{\prec_j}\), which implies \(w(G, C) \leq \text{card} J = m\) and we have \(w(G, C) = m = \sup \{w(G_i, C_i); i \in I\}\). The assertion on \(W(G, C)\) follows from 2.7.

3. COCYCLICALLY ORDERED SETS

3.1. Definition. Let \(G\) be a set, \(T\) a ternary relation on \(G\). \(T\) is called a cocyclic order on \(G\) if it is

\(\text{(v) reflexive, i.e. } x, y, z \in G, \text{ card } \{x, y, z\} \leq 2 \Rightarrow (x, y, z) \in T,\)

cyclic, complete and satisfies the condition

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(vi) $x, y, z, u \in G$, pairwise distinct, $(x, y, z) \in T \Rightarrow (x, y, u) \in T$ or $(x, u, z) \in T$.

If $G$ is a set and $T$ a cocyclic order on $G$, then the pair $(G, T)$ is called a cocyclically ordered set.

If $G$ is a set and $T$ a ternary relation on $G$, then we denote by $\text{Co } T$ the complement of $T$ in $G^3$, i.e. $\text{Co } T = G^3 - T$.

3.2. Theorem. Let $G$ be a set, $T$ a ternary relation on $G$. $T$ is a cocyclic order on $G$ iff $\text{Co } T$ is a cyclic order on $G$.

Proof. 1. Let $T$ be a cocyclic order on $G$ and denote $\text{Co } T = C$. Assume that there exist $x, y, z \in G$ with $(x, y, z) \in C$, $(z, y, x) \in C$. Then $(x, y, z) \in T$, $(z, y, x) \in T$ which implies $x \not= y \not= z \not= x$ and thus $T$ is not complete. This is a contradiction and hence $C$ is asymmetric. Let $(x, y, z) \in C$ and assume $(y, z, x) \in C$. Then $(y, z, x) \in T$ and as $T$ is cyclic, $(x, y, z) \in T$, a contradiction. Thus $C$ is cyclic. Let $(x, y, z) \in C$, $(x, z, u) \in C$. Then $x \not= y \not= z \not= x$, $x \not= z \not= u \not= x$ and we shall show $y \not= u$. If $y = u$, then $(x, z, y) \in C$, thus $(z, y, x) \in C$ as $C$ is cyclic and this contradicts the asymmetry of $C$. Thus the elements $x, y, z, u$ are pairwise distinct. Assume $(x, y, u) \in C$. Then $(x, y, u) \in T$ and, by (vi), either $(x, y, z) \in T$ or $(x, z, u) \in T$. But this contradicts the assumption $(x, y, z) \in C$, $(x, z, u) \in C$. We have shown that $C$ is transitive and thus $C = \text{Co } T$ is a cyclic order on $G$.

2. Let $C = \text{Co } T$ be a cyclic order on $G$. From the asymmetry and cyclicity of $C$ we easily derive $(x, y, z) \in C \Rightarrow x \not= y \not= z \not= x$. Thus $x, y, z \in G$, $\text{card } \{x, y, z\} \leq 2 \Rightarrow (x, y, z) \in C$, hence $(x, y, z) \in T$ and the relation $T$ is reflexive. Let $(x, y, z) \in T$ and assume $(y, z, x) \in T$. Then $(y, z, x) \in C$ and by the cyclicity of $C$, $(x, y, z) \in C$ which is a contradiction. Thus $T$ is cyclic. Let $x, y, z \in G$, $x \not= y \not= z \not= x$ and assume $(x, y, z) \in T$, $(z, y, x) \in T$. Then $(x, y, z) \in C$, $(z, y, x) \in C$, which contradicts the asymmetry of $C$. Hence $T$ is complete. Let $x, y, z, u \in G$ be pairwise distinct elements such that $(x, y, z) \in T$ and assume $(x, y, u) \in T$, $(x, u, z) \in T$. Then $(x, y, u) \in C$, $(x, u, z) \in C$ and hence $(x, y, z) \in C$ by the transitivity of $C$, which is a contradiction. Thus $(x, y, u) \in T$ or $(x, u, z) \in T$, $T$ satisfies (vi) and is, therefore, a cocyclic order on $G$.

3.3 Corollary. Let $G$ be a set, $<_i$ an order on $G$. Then $\text{Co } C_{<_i}$ is a cocyclic order on $G$.

3.4. Theorem. Let $G$ be a set, $(<_i)_{i \in I}$ a family of orders on $G$. Then $\bigcap_{i \in I} \text{Co } C_{<_i}$ is a cocyclic order on $G$ iff the family $(<_i)_{i \in I}$ is harmonized.

Proof. Clearly $\bigcap_{i \in I} \text{Co } C_{<_i} = \text{Co } \left( \bigcup_{i \in I} C_{<_i} \right)$ so that — by 3.2 — $\bigcap_{i \in I} \text{Co } C_{<_i}$ is a cocyclic order on $G$ iff $\bigcup_{i \in I} C_{<_i}$ is a cyclic order on $G$. But this holds by 1.8 iff the family $(<_i)_{i \in I}$ is harmonized.
3.5. Theorem. Let \((G, T)\) be a cocyclically ordered set. Then there exists a harmonized family \((<\rangle)_{i \in I}\) of orders on \(G\) such that \(T = \bigcap_{i \in I} \text{Co} C_{<i}\).

Proof. As \(\text{Co} T\) is a cyclic order on \(G\), by 1.9 there exists a harmonized family \((<\rangle)_{i \in I}\) of orders on \(G\) such that \(\text{Co} T = \bigcup_{i \in I} C_{<i}\). But then \(T = \bigcap_{i \in I} \text{Co} C_{<i}\).

Analogously, from 1.10 we obtain \(T = \bigcap_{i \in I} \text{Co} C_{<i}\).

3.6. Corollary. Let \((G, T)\) be a cocyclically ordered set. Then there exists a harmonized family \((<\rangle)_{i \in I}\) of linear orders in \(G\) such that \(T = \bigcap_{i \in I} \text{Co} C_{<i}\).

3.7. Definition. Let \((G, T)\) be a cocyclically ordered set. Put \(d(G, T) = \min \{\text{card} I; \text{there exists a harmonized family } (<\rangle)_{i \in I}\text{ of orders on } G\text{ such that } T = \bigcap_{i \in I} \text{Co} C_{<i}\}\), \(D(G, T) = \min \{\text{card} I; \text{there exists a harmonized family } (<\rangle)_{i \in I}\text{ of linear orders in } G\text{ such that } T = \bigcap_{i \in I} \text{Co} C_{<i}\}\).

3.8. Theorem. Let \((G, T)\) be a cocyclically ordered set. Then \(d(G, T) = w(G, \text{Co} T), D(G, T) = W(G, \text{Co} T)\).

Proof. For any harmonized family \((<\rangle)_{i \in I}\) of orders on \(G\) the relation \(T = \bigcap_{i \in I} \text{Co} C_{<i}\) is equivalent to the relation \(\text{Co} T = \bigcup_{i \in I} C_{<i}\). This yields both the assertions.

References


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