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ISOMETRIES OF MULTILATTICE GROUPS

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Isometries in abelian lattice ordered groups were studied by K. L. Swamy [10], [11] and W. B. Powell [9]; the non-abelian case was dealt with in the papers [4], [5]. J. Trías [12] developed the theory of isometries in Riesz spaces.
Multilattice groups were introduced by M. Benado [1]. A thorough investigation of multilattice groups was performed by McAllister [7], [8]. In the present paper it will be shown that the results on the relations between isometries and direct decompositions of lattice ordered groups [4] can be extended to hold for abelian distributive multilattice groups.

1. PRELIMINARIES

At first we recall some notions concerning multilattices and multilattice groups. Let $P$ be a partially ordered set. If $a$ and $b$ are elements of $P$, then we denote by $U(a, b)$ and $L(a, b)$ the set of all upper bounds or the set of all lower bounds of the set $\{a, b\}$, respectively. Next we denote by $a \bigvee_m b$ the set of all minimal elements of the set $U(a, b)$; analogously, $a \bigwedge_m b$ is defined to be the set of all maximal elements of the set $L(a, b)$.

The partially ordered set $P$ is said to be a multilattice (Benado [1]) if it fulfils the conditions for each pair $a, b \in P$:

(m$_1$) If $x \in U(a, b)$, then there is $x_1 \in a \bigvee_m b$ such that $x_1 \leq x$.

(m$_2$) If $y \in L(a, b)$, then there is $y_1 \in a \bigwedge_m b$ such that $y_1 \geq y$.

A multilattice $P$ is called distributive if, whenever $a, b, c$ are elements of $P$ such that

$$(a \bigwedge_m b) \cap (a \bigwedge_m c) \neq \emptyset$$

and

$$(a \bigvee_m b) \cap (a \bigvee_m c) \neq \emptyset,$$

then $b = c$. (See [1] and [7].)

For the basic notions and denotations concerning partially ordered groups and lattice ordered groups cf. Fuchs [3] and Conrad [2]. The group operation in partially ordered groups will be written additively.
Let $G$ be a partially ordered group such that (i) $G$ is directed, and (ii) the partially ordered set $(G; \leq)$ is a multilattice. Then $G$ is called a multilattice group. All multilattice groups dealt with in this paper are assumed to be abelian.

If $G$ is a lattice ordered group and $x \in G$, then we can define the absolute value $|x|$ in several equivalent ways; e.g., we can put

$$(1) \quad |x| = 2z - x,$$

where $z = x \lor 0$.

Now let $G$ be a multilattice group and let $x \in G$. Using an analogy with (1) we define

$$(1') \quad |x| = \{2z - x : z \in x \lor_m 0\}.$$

Hence $|x|$ is a nonempty set for each $x \in G$. If $|x| = \{y\}$ is a one-element set, then we write also $|x| = y$. In the case $x \geq 0 \ (x \leq 0)$ we have $|x| = x \ (|x| = -x)$.

Let $f$ be a one-to-one mapping of $G$ onto $G$ such that the relation

$$(\alpha) \quad |f(x) - f(y)| = |x - y|$$

is valid for each $x \in G$ and $y \in G$. Then $f$ is said to be an isometry of $G$.

If $f$ is an isometry of $G$ and $f(0) = 0$, then $f$ will be called a 0-isometry. Let $a \in G$; the mapping $f_a$ of $G$ onto $G$ defined by $f_a(x) = x + a$ for each $x \in G$ is a translation on $G$. Every translation is an isometry on $G$. Each isometry can be uniquely represented as a composition of a 0-isometry and a translation. Hence for determining all isometries of $G$ it suffices to find all 0-isometries.

2. REGULAR QUADRUPLES

Let $G$ be a multilattice group. A quadruple $\{a, b, u, v\}$ of elements of $G$ is said to be regular if $u \in a \lor_m b$, $v \in a \lor_m b$ and $v - a = b - u$.

2.1. Lemma. Let $a, b \in G$, $v \in a \lor_m b$. Put $u = a + b - v$. Then $\{a, b, u, v\}$ is a regular quadruple.

Proof. It suffices to verify that $u \in a \lor_m b$. We have $0 \leq v - a = b - u$, hence $b \geq u$, and analogously $a \geq u$. There exists $u_1 \in a \lor_m b$ with $u \leq u_1$. Let $u', a', b' \in G$ such that $u_1 = u + u'$, $a = u_1 + a'$, $b = u_1 + b'$. Then $u' \geq 0$, $a' \geq 0$ and $b' \geq 0$. Because of $a - u = u' + a'$, $b - u = u' + b'$ we obtain $v - u' = b + a'$, $v - u' = a + b'$, hence $v - u' \in U(a, b)$. Therefore $u' = 0$ and thus $u = u_1 \in a \lor_m b$.

The assertion dual to 2.1 can be proved analogously.

2.2. Lemma. Let $\{a, b, u, v\}$ be a regular quadruple. Let $a_1 \in [u, a]$, $b_1 = a + a_1 - u$. Then $\{a_1, b, u, b_1\}$ and $\{a, b_1, a_1, v\}$ are regular quadruples.
Proof. From 2.1 it follows that \( \{a, b_1, a_1, v\} \) is a regular quadruple. Next, from the assertion dual to 2.1 we infer that \( \{a_1, b, u, b_1\} \) is a regular quadruple.

2.3. Lemma. Let \( \{a, b, u, v\} \) be a regular quadruple in \( G \), \( 0 \leq p \leq a - u, x \in [u + p, b + p] \). Put \( x - (u + p) = q \). Then \( \{a, x, u + p, a + q, b, x, u + q, b + p\}, \{u + p, u + q, u, x\} \) and \( \{a + q, b + p, x, v\} \) are regular quadruples.

Proof. This is a consequence of 2.2.

![Fig. 1.](image)

Again, let \( \{a, b, u, v\} \) be a regular quadruple in \( G \). Assume that \( x \in [u, v] \). Let us apply the following construction (cf. Fig. 1).

We choose \( a_1 \in a \wedge_m x \) with \( a_1 \geq u \). Denote \( b'_2 = b + (a_1 - u) \). In view of 2.2, \( \{a_1, b, u, b'_2\} \) and \( \{a, b'_2, a_1, v\} \) are regular quadruples.

Choose \( u_1 \in b'_2 \wedge_m x \) with \( u_1 \geq a_1 \). Denote \( b_1 = u + (u_1 - a_1), a'_2 = a + (u_1 - a_1) \). According to 2.3, the quadruples

\[
\{a_1, b_1, u, u_1\}, \{a, u_1, a_1, a'_2\}, \{u_1, b, b_1, b'_2\}, \{a'_2, b'_2, u_1, v\}
\]

are regular.

2.4. Lemma. \( u_1 \in a'_2 \wedge_m x \).

Proof. We have \( u_1 \leq x \) and \( u_1 \leq a'_2 \). Hence there is \( z \in a'_2 \wedge_m x \) with \( z \geq u_1 \). If \( z > u_1 \), then we should have

\[
a_1 < a_1 + (z - u_1) \leq u_1 + (z - u_1) = z \leq x,
a_1 + (z - u_1) \leq a_1 + (a'_2 - u_1) = a_1 + (a - a_1) = a,
\]

604
hence \( a_i \notin a \land_m x \), which is a contradiction. Therefore \( z = u_1 \), completing the proof of the lemma.

The further steps of our construction are dual to the previous ones with the distinction that we consider the regular quadruple \( \{a_2', b_2', u_1, v\} \) instead of \( \{a, b, u, v\} \).

We choose \( a_2 \in a_2' \lor_m x \) with \( a_2 \leq v \). Denote \( b' = b_2' + (a_2 - v) \). In view of 2.2, \( \{a_2', b_2', u_1, a_3\} \) and \( \{a_2, b_2', b', v\} \) are regular quadruples. Put \( b_1' = b + (a_2 - v) \); then according to 2.3 the quadruples \( \{u_1, b_1', b_1, b'\} \) and \( \{b', b, b_1', b_2\} \) are regular as well.

Now choose \( v_1 \in b' \lor_m x \) with \( v_1 \leq a_2 \). Denote \( b_2 = v + (v_1 - a_2) \), \( a' = a_2' + + (v_1 - a_2) \). In view of 2.3, all the quadruples \( \{a', b', u_1, v_1\}, \{a_2', v_1, a', a_2\}, \{v_1, b_2', b', b_2\} \) and \( \{a_2, b_2, v_1, v\} \) are regular. Put \( a_1' = a + (v_1 - a_2) \). Then according to 2.2, the quadruples \( \{a_1', u_1, a_1, a'\} \) and \( \{a, a', a_1', a_2'\} \) are regular as well.

By an argument dual to that applied in the proof of 2.4 we obtain

\[ v_1 \in a' \lor_m x \]

We shall prove that the equivalence

\[
(*) \quad a_2' = a_2 \iff a' = b'
\]

is valid.

Let \( a_2' = a_2 \) hold. Since \( \{a_2', v_1, a', a_2\} \) is a regular quadruple, we infer that \( a' = v_1 \), hence \( x \leq a' \) and thus \( a' \land_m x = \{x\} \). In view of 2.4 we have \( u_1 \in a' \land_m x \), hence \( u_1 = x \). From the fact that \( \{a', b', u_1, v_1\} \) is a regular quadruple we obtain that \( u_1 = b' \). Thus \( b' \lor_m x = x \lor_m x = \{x\} \); because of \( v_1 \in b' \lor_m x \) we have \( v_1 = x \) and so \( u_1 = v_1 \), implying \( a' = b' \).

Conversely, assume that \( a' = b' \). Then \( a' \lor_m b' = \{a'\} \) hence \( v_1 = a' \). Since \( \{a_2', v_1, a', a_2\} \) is regular, we infer that \( a_2 = a_2' \) holds.

Similarly we can verify that the relation \( a' = b' \) is equivalent to each of the following relations: \( b_2' = b_2; a_1 = a_1' \); \( b_1 = b_1' \).

2.5. Lemma. If \( a_2 < a_2' \), then \( G \) fails to be distributive.

Proof. This follows from \((*)\) and from the definition of distributivity (cf. Sec. 1).

2.6. Lemma. Assume that \( (G; \geq) \) is distributive. Let \( \{a, b, u, v\} \) be a regular quadruple in \( G \), \( x \in [u, v] \) and \( a_1 \in a \land_m x \), \( a_1 \geq u \). Then there are elements \( b_1 \in [u, b], \ a_2 \in [a, v] \) and \( b_2 \in [b, v] \) such that \( \{a, x, a_1, a_2\}, \ \{b, x, b_1, b_2\}, \ \{a_1, b_1, u, x\} \) and \( \{a_2, b_2, x, v\} \) are regular quadruples.

Proof. Let \( a_2, b_1 \) and \( b_2 \) be as in the construction above. In view of 2.5 we have \( a_2 = a_2' \); similarly, the relations \( b_2' = b_2; a_1 = a_1' \) and \( b_1 = b_1' \) hold. Hence all the quadruples involved in the assertion of the lemma are regular.
3. AUXILIARY RESULTS ON ISOMETRIES

In this section we assume that $G$ is a distributive multilattice group and $f$ is an isometry on $G$.

Let $x, y, z \in G$, $t = z + y$. The relation $z \in 0 \vee_m (x - y)$ is equivalent to $t \in x \vee_m y$, whence

\[ |x - y| = \{2t - x - y : t \in x \vee_m y \}. \]

By using 2.1 and the assertion dual to 2.1 we obtain also

\[ |x - y| = \{x + y - 2z : z \in x \land_m y \}. \]

3.1. **Lemma.** Let $a, b, x \in G$, $a \leq x \leq b$. Assume that $f(a) \leq f(b)$. Then $f(a) \leq f(x) \leq f(b)$.

**Proof.** We have $|b - x| = b - x$, $|x - a| = x - a$, hence in view of (2)(cf. Sec. 1) $|f(b) - f(x)|$ and $|f(x) - f(a)|$ are one-element sets. Choose $u \in f(a) \land_m f(x)$, $v \in f(b) \land_m f(x)$. In view of (3.1) and (3.2) we obtain

\[ |f(b) - f(x)| = 2v - f(b) - f(x), \]
\[ |f(x) - f(a)| = f(x) + f(a) - 2u. \]

Because of

\[ |b - a| = |b - x| + |x - b| \]

we obtain

\[ |f(b) - f(a)| = |f(b) - f(x)| + |f(x) - f(b)|, \]

hence

\[ f(b) - f(a) = 2v - f(b) - f(x) + f(x) + f(a) - 2u = \]
\[ = (v - f(b)) + (v - u) + (f(a) - u) \geq v - u. \]

We evidently have $v - u \geq f(b) - f(a)$. Thus $v - u = f(b) - f(a)$ and (since $v - f(b) \geq 0$, $f(a) - u \geq 0$) we get $v = f(b)$, $u = f(a)$. Hence $f(a) \leq f(x) \leq f(b)$. Analogously we can verify

3.1.1. **Lemma.** Let $a, b, x \in G$, $a \leq x \leq b$. Assume that $f(a) \geq f(b)$. Then $f(a) \geq f(x) \geq f(b)$.

3.2. **Lemma.** Let $x, y \in G$, $x \geq 0 \geq y$. If $f(x) \geq 0$, then $f(x) = x$. If $f(x) \leq 0$, then $f(x) = -x$. If $f(y) \geq 0 (f(y) \leq 0)$, then $f(y) = -y (f(y) = y)$.

**Proof.** Let $f(x) \geq 0$. Then $x = |x| = |f(x)| = f(x)$. The other assertions can be verified analogously.

3.3. **Lemma.** Let $x, y \in G$, $x \geq y$, $u \in f(x) \land_m f(y)$, $u = f^{-1}(u')$. Then $u \in [y, x]$.
Proof. From (3.2) we infer \( f(x) + f(y) - 2u' \in |f(x) - f(y)| \). Since \( |f(x) - f(y)| = |x - y| = x - y \), we have card \( |f(x) - f(y)| = 1 \), whence

\[
|f(x) - f(y)| = f(x) + f(y) - 2u' = f(x) - f(u) + f(y) - f(u) = |f(x) - f(u)| + |f(y) - f(u)|.
\]

Both \( |f(x) - f(u)| \) and \( |f(u) - f(y)| \) are one-element sets. Hence

\[
|x - y| = |x - u| + |u - y|
\]

and both \( |x - u| \) and \( |u - y| \) are one-element sets. Choose \( u_1 \in y \Delta_m u, v_1 \in x \Delta_m u \).

Then

\[
|x - u| = 2v_1 - x - u, \quad |u - y| = u + y - 2u_1,
\]

whence

\[
x - y = |x - y| = (v_1 - u_1) + (v_1 - x) + (y - u_1) \geq v_1 - u_1.
\]

Because of \( u_1 \leq y \leq x \leq v_1 \) we have \( v_1 - u_1 \geq x - y \), therefore \( x - y = v_1 - u_1 \) and thus \( v_1 = x, u_1 = y \). Hence \( y \leq u \leq x \).

Similarly we obtain:

3.4. Lemma. Let \( x, y \in G, y \leq x, v' \in f(x) \Delta_m f(y), v = f^{-1}(v') \). Then \( v \in [y, x] \).

3.5. Lemma. Let \( x, y, u, v \) be as in 3.3 and 3.4. Then \( y \in u \Delta_m v, x \in u \cup_m v \).

Proof. Let \( u_1 \in u \Delta_m v, y \leq u_1 \). Since \( y \leq u_1 \leq u \) and \( f(y) \geq f(u) \), according to 3.1.1 we have \( f(y) \geq f(u_1) \). On the other hand, from 3.1 and from the relations \( y \leq u_1 \leq v, f(y) \leq f(v) \) we obtain \( f(y) \leq f(u_1) \). Thus \( f(u_1) = y \), hence \( y \in u \Delta_m v \).

Analogously we can prove that \( x \in u \cup_m v \).

In the above consideration, \( v' \) was an arbitrary element of the set \( f(x) \Delta_m f(y) \). Now assume that \( \{f(x), f(y), u', v'\} \) is a regular quadruple. Such an element \( v' \) does exist (cf. the dual of 2.1). Under this assumption we have:

3.6. Lemma. \( \{u, v, y, x\} \) is a regular quadruple.

Proof. In view of 3.5 we have to verify that \( x - v = u - y \). In fact,

\[
|x - v| = |f(x) - f(v)| = f(v) - f(x) = f(y) - f(u) = |f(y) - f(u)| =
\]

\[
|y - u| = u - y.
\]

3.7. Lemma. Let \( x, y, u, v \) be as in 3.6. Let \( z \in [y, x] \) and assume that \( f(z) \leq f(y) \). Then \( z \leq u \).

Proof. From 2.6 it follows that there are elements \( u_1 \in [y, u] \) and \( v_1 \in [y, v] \) such that \( z \in u_1 \cup_m v_1 \). In view of 3.1.1 and 3.1 we have \( f(v_1) \geq f(y) \) (since \( f(v) \geq f(y) \)) and at the same time, \( f(v_1) \leq f(y) \) (since \( f(z) \leq f(y) \); thus \( f(v_1) = f(y) \)). Therefore \( v_1 = y \) and thus \( z = u_1 \leq u \).

Analogously we obtain:
3.8. Lemma. Let \( x, y, u, v \) be as in 3.6. Let \( z \in [y, x] \) and assume that \( f(z) \geq f(y) \). Then \( z \leq v \).

3.9. Lemma. Let \( x, y \in G, x \geq y \). Then both \( f(x) \wedge_m f(y) \) and \( f(x) \vee_m f(y) \) are one-element sets.

Proof. Let us apply the above denotations. Let \( u'' \in f(x) \wedge_m f(y) \). In view of 3.3 we have \( f^{-1}(u'') \in [y, x] \) and thus, according to 3.7, \( f^{-1}(u'') \leq f^{-1}(u') \). But the roles of \( u' \) and \( u'' \) can be interchanged, whence \( f^{-1}(u') \leq f^{-1}(u'') \). Therefore \( u'' = u' \) and hence \( \text{card}(f(x) \wedge_m f(y)) = 1 \). In view of the assertion dual to 2.1 we infer that \( f(x) \vee_m f(y) \) is a one-element set as well.

3.10. Corollary. Let \( x, y \in G, x \geq y \). Then the elements \( u, v \) from 3.3 and 3.4 are uniquely determined.

Now let \( 0 \leq x \in G; \) put \( y = 0 \). Let \( u, v \) be as above.

In view of 3.10 we denote \( u = x_u, v = x_v \). Since \( \{u_x, v_x, 0, x\} \) is a regular quadruple (cf. 3.6) we have \( x = x_u + x_v \).

4. THE SETS \( A \) AND \( B \)

Again, let \( G \) be a distributive multilattice and let \( f \) be an isometry of \( G \) with \( f(0) = 0 \). We denote

\[
A_1 = \{x \in G : x \geq 0 \text{ and } f(x) \geq 0\},
\]

\[
B_1 = \{x \in G : x \geq 0 \text{ and } f(x) \leq 0\}.
\]

4.1. Lemma. The set \( A_1 \) is closed with respect to the operation \( + \).

Proof. Let \( a_1, a_2 \in A_1, x = a_1 + a_2, u = x_u, v = x_v \). In view of 3.8 we have

\[
a_1 \leq v, \quad a_2 \leq v.
\]

Because of \( x \leq 2v \) and \( x = u + v \), the relation \( u \leq v \) is valid. According to 3.5, \( 0 \in u \wedge_m v \); hence \( u = 0 \). Therefore \( 0 = f(u) \leq f(x) \), yielding \( x \in A_1 \).

Analogously we can verify

4.2. Lemma. The set \( B_1 \) is closed with respect to the operation \( + \).

4.3. Lemma. Let \( a \in A_1, b \in B_1, x = a + b \). Then \( x_u = a \) and \( x_v = b \).

Proof. From 3.7 and 3.8 we obtain \( 0 \leq a \leq x_u, 0 \leq b \leq x_v \), hence

\[
x = a + b \leq x_u + x_v = x.
\]

Thus we must have \( a = x_u, b = x_v \).

From 4.1, 4.2 and 4.3 we obtain:

608
4.4. Lemma. Let \( x, y \in G, \ x \geq 0, \ y \geq 0 \). Then \( (x + y)_u = x_u + y_u, \ (x + y)_v = x_v + y_v \).

4.5. Lemma. Let \( x \) and \( y \) be as in 4.4. Then the following conditions are equivalent: (i) \( y \leq x \); (ii) \( y_u \leq x_u \) and \( y_v \leq x_v \).

Proof. The implication (ii) \( \Rightarrow \) (i) is obvious. The implication (i) \( \Rightarrow \) (ii) follows from 3.3, 3.4, 3.7 and 3.8.

4.6. Lemma. The partially ordered semigroup \( G^+ = \{ g \in G : g \geq 0 \} \) is a direct product of the partially ordered semigroups \( A_1 \) and \( B_1 \).

Proof. This is a consequence of 4.4 and 4.5.

Put \( A = A_1 - A_1, \ B = B_1 - B_1 \). From 4.6 and Thm. 2.3 [6] we infer:

4.7. Lemma. The partially ordered group \( G \) is a direct product of partially ordered groups \( A \) and \( B \).

4.7.1. Remark. For \( g \in G \) we denote by \( g_A \) and \( g_B \) the component of \( g \) in the direct factor \( A \) and \( B \), respectively. If \( 0 \leq x \in G \) and \( u, v \) are as in 3.10 (with \( y = 0 \)), then according to the definition of \( A_1 \) and \( B_1 \) we have

\[ x_A = u, \quad x_B = v. \]

4.8. Lemma. Let \( \{a, b, u, v\} \) be a regular quadruple. Assume that \( f(a) \leq f(u), \ f(a) \leq f(v) \). Then \( \{f(u), f(v), f(a), f(b)\} \) is a regular quadruple.

Proof. From 3.1 we obtain (by considering the isometry \( f^{-1} \)) that \( f(a) \in f(u)_m \Lambda_m f(b) \) holds. In view of 3.10, \( f(u) \Lambda_m f(b) \) is a one-element set, hence \( f(u)_m f(b) = \{f(a)\} \). Also (see 3.10), \( f(u) \Lambda_m f(b) \) is a one-element set; let us write \( f(u) \Lambda_m f(b) = \{f(v_1)\} \). Then the quadruple \( \{f(u), f(v), f(a), f(v_1)\} \) must be regular. Now from 3.6 it follows that \( \{a, v_1, u, v\} \) is a regular quadruple, thus \( v_1 = b \).

4.9. Lemma. For each \( x \in G \) we have \( f(x) = x_A - x_B \).

Proof. Chose \( u \in 0 \Lambda_m x \). According to the dual of 2.1 there exists \( v \in 0 \Lambda_m x \) such that \( \{0, x, u, v\} \) is a regular quadruple. Then \( x = u + v \), hence

\[ x_A = u_A + v_A, \quad x_B = u_B + v_B. \]

In view of 3.3, 3.4 and 3.2 there exist \( u_1 \in [u, 0], \ v_1 \in [0, v] \) such that

\[ 0 = f(0) \leq f(v_1) = v_1, \quad f(v_1) \geq f(v), \]

\[ 0 \leq f(u_1) = -u_1, \quad f(u_1) \geq f(u). \]

609
(Cf. Fig. 2; dashed lines denote the fact that the corresponding interval is reversed under \( f \) e.g. \( u_1 < 0 \) and \( f(u_1) > f(0) \).)

\[
f(u_1) \leq f(z), \quad f(v_1) \leq f(z).
\]

Put \( t = v + u_1 \). In view of 2.6 we have \( z \leq t \), and clearly \( t \leq v \). Since \( f(z) \geq f(v) \), from 3.1 it follows

\[
f(z) \geq f(t) \geq f(v).
\]

Next we put \( t' = u + v_1 \). In view of 2.6 we have

\[
u \leq t' \leq z.
\]

Because of \( f(u) \leq f(z) \), by using 3.1 we get

\[
f(u) \leq f(t') \leq f(z).
\]

From 2.6 it follows that \( \{z, x, t', t\} \) is a regular quadruple. In view of the dual to 4.8 we obtain that

\[
f(t') \geq f(x), \quad f(x) \leq f(t).
\]

By applying the above inequalities we infer

\[
f(x) = (f(x) - f(t')) + f(t') - (f(u)) + (f(u) - f(u_1)) + (f(u_1) - f(0)) = \\
= -|f(x) - f(t')| + |f(t') - f(u)| - |f(u) - f(u_1)| + \\
+ |f(u_1) - f(0)| = -|x - t'| + |t' - u| - |u - u_1| + |u_1 - 0| =
\]
\[-(x - t') + (t' - u) + (u - u_1) - (u_1 - 0) =
= -(v - v_1) + (v_1 - 0) + (u - u_1) - u_1.\]

According to 4.7.1 we have \(v_1 = v_A\), hence \(v - v_1 = v_B\). Similarly we have \(u_1 = u_B\), hence \(u - u_1 = u_A\). Thus
\[f(x) = -v_B + v_A + u_A - u_B = (u + v)_A - (u + v)_B = x_A - x_B.\]

Let \(G = P \times Q\) be any direct decomposition of \(G\). Then for arbitrary \(x, y \in G\) we have
\[x \land_m y = (x_p \land_m y_p) + (x_q \land_m y_q),\]
and analogously for \(\lor_m\). From this we obtain
\[|x| = |x_p| + |x_q|.

**4.10. Lemma.** Let \(G = P \times Q\). For each \(x \in G\) define \(g(x) = x_p - x_Q\). Then \(g\) is an isometry of \(G\) and \(g(0) = 0\).

**Proof.** Let \(x, y \in G\). Then \(g(x - y) = g(x) - g(y)\). Thus
\[|g(x) - g(y)| = |g(x - y)| = |(g(x - y))_p| + |(g(x - y))_q| =
= |(x - y)_p| + |(x - y)_q| = |x - y|.

Clearly \(g(0) = 0\).

Summarizing, we have

**4.11. Theorem.** Let \(G\) be a distributive abelian multilattice group. For each isometry \(f\) on \(G\) with \(f(0) = 0\) there exist a direct decomposition \(G = A \times B\) such that \(f(x) = x_A - x_B\) is valid for each \(x \in G\). Conversely, if \(G = P \times Q\) is a direct decomposition of \(G\) and if we put \(g(x) = x_p - x_Q\) for each \(x \in G\), then \(g\) is an isometry on \(G\) with \(g(0) = 0\).

The question whether the assumption of distributivity or commutativity of \(G\) in the above theorem can be cancelled remains open.

The first author announces the following correction to the paper [4] concerning isometries of \(l\)-groups: In the assertion (**) of § 3 it should be assumed that \(B_0(G)\) is the system of all abelian direct factors of \(G\) and that \(B \in B_0(G)\). Theorem 2.5, which is the main result of [4], remains unchanged. The author is indebted to A. M. W. Glass for this observation.

**References**


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