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DIMENSION THEORY FOR CYCLICALLY AND COCYCLICALLY ORDERED SETS

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In [6], a general dimension theory for algebraic structures is developed. In this note we show that this theory can be applied also to cyclically and cocyclically ordered sets; the dimension of these sets — with a suitable choice of basic classes — coincides with the characteristics w, W, d, D as introduced in [7].

1. K_4 -CLASSES

Here we summarize some necessary concepts from [6].

1.1. Notation. Let $I \neq \emptyset$ be a set, $(G_i; i \in I)$ an indexed family of sets, i.e. a mapping assigning a set G_i to any $i \in I$. We put $\sum_{i \in I} G_i = \{(i, k); i \in I, k \in G_i\}$. Let \mathcal{C} be a class; an indexed family $(G_i; i \in I)$ with $G_i \in \mathcal{C}$ is called a *family of elements in \mathcal{C}* . By $\mathcal{S}(\mathcal{C})$ we denote the class of all families of elements in \mathcal{C} .

1.2. Definition. Let \mathcal{C} be a class, R a correspondence between \mathcal{C} and $\mathcal{S}(\mathcal{C})$ with the property: $G, H \in \mathcal{C}, (H_i; i \in I) \in \mathcal{S}(\mathcal{C}), G R(H), H R(H_i; i \in I) \Rightarrow G R(H_i; i \in I)$. Then the pair (\mathcal{C}, R) is called a K_1 -class.

1.3. Notation. Let (\mathcal{C}, R) be a K_1 -class, $\mathcal{L} \subseteq \mathcal{C}$. We denote $\mathcal{C}(\mathcal{L}, R) = \{G \in \mathcal{C}; \text{there exists } (G_i; i \in I) \in \mathcal{S}(\mathcal{L}) \text{ with } G R(G_i; i \in I)\}$.

1.4. Definition. Let (\mathcal{C}, R) be a K_1 -class, $\mathcal{L} \subseteq \mathcal{C}$. For any $G \in \mathcal{C}(\mathcal{L}, R)$ put $(\mathcal{L}, R) - \dim G = \min \{\text{card } I; \text{there exists } (G_i; i \in I) \in \mathcal{S}(\mathcal{L}) \text{ with } G R(G_i; i \in I)\}$. The cardinal $(\mathcal{L}, R) - \dim G$ is called the (\mathcal{L}, R) -dimension of G .

1.5. Definition. Let \mathcal{C} be a class, $T: \mathcal{S}(\mathcal{C}) \rightarrow \mathcal{C}$ a partial mapping with the property: $I = \{i_0\}, G_{i_0} = G \in \mathcal{C} \Rightarrow (G_i; i \in I) \in \text{dom } T$ and $TG_i = G$. Let \preceq be a (fixed) preorder on \mathcal{C} . Then the triple $(\mathcal{C}, T, \preceq)$ is called a K_2 -class.

1.6. Remark. Any K_2 -class (\mathcal{C}, T, \leq) is a K_1 -class if we define a correspondence $R(T, \leq)$ between \mathcal{C} and $\mathcal{S}(\mathcal{C})$ by $GR(T, \leq) (G_i; i \in I) \leftrightarrow (G_i; i \in I) \in \text{dom } T$ and $G \leq \prod_{i \in I} TG_i$.

1.7. Definition. Let (\mathcal{C}, T, \leq) be a K_2 -class with the following properties:
 (1) if $I \neq \emptyset$ is a set, $K_i \neq \emptyset$ a set for any $i \in I$, $G_{i,k} \in \mathcal{C}$ for any $(i, k) \in \sum_{i \in I} K_i$ and $(G_{i,k}; (i, k) \in \sum_{i \in I} K_i) \in \text{dom } T$, $(G_{i,k}; k \in K_i) \in \text{dom } T$ for any $i \in I$, $(\prod_{i \in I} TG_{i,k}; i \in I) \in \text{dom } T$, then $\prod_{i \in I} TG_{i,k} = T(\prod_{i \in I} \prod_{k \in K_i} TG_{i,k})$ (the associative rule)
 (2) if $I \neq \emptyset$ is a set, $(G_i; i \in I) \in \text{dom } T$, $(H_i; i \in I) \in \text{dom } T$ and $G_i \leq H_i$ for any $i \in I$, then $\prod_{i \in I} TG_i \leq \prod_{i \in I} TH_i$.

Then (\mathcal{C}, T, \leq) is called a K_3 -class. If (\mathcal{C}, T, \leq) is a K_3 -class and $\text{dom } T = \mathcal{S}(\mathcal{C})$, then (\mathcal{C}, T, \leq) is called a K_4 -class.

1.8. Remark. If (\mathcal{C}, T, \leq) is a K_4 -class and $\mathcal{L} \subseteq \mathcal{C}$, we write $\mathcal{C}(\mathcal{L}; T, \leq)$ instead of $\mathcal{C}(\mathcal{L}, R(T, \leq))$, and $(\mathcal{L}; T, \leq) - \dim G$ instead of $(\mathcal{L}, R(T, \leq)) - \dim G$. Thus, $\mathcal{C}(\mathcal{L}; T, \leq) = \{G \in \mathcal{C}; \text{there exists } (G_i; i \in I) \in \mathcal{S}(\mathcal{L}) \text{ with } G \leq \prod_{i \in I} TG_i\}$ and $(\mathcal{L}; T, \leq) - \dim G = \min \{\text{card } I; \text{there exists } (G_i; i \in I) \in \mathcal{S}(\mathcal{L}) \text{ with } G \leq \prod_{i \in I} TG_i\}$ for any $G \in \mathcal{C}(\mathcal{L}; T, \leq)$.

1.9. Notation. Let \mathcal{C} be the class of all pairs (G, C) where G is a set and C a ternary relation on G . Let us put $\bigcup_{i \in I} (G_i, C_i) = (\bigcup_{i \in I} G_i, \bigcup_{i \in I} C_i)$, $\bigcap_{i \in I} (G_i, C_i) = (\bigcap_{i \in I} G_i, \bigcap_{i \in I} C_i)$ for any $((G_i, C_i); i \in I) \in \mathcal{S}(\mathcal{C})$. Let $\text{id}_{\mathcal{C}}$ be the identity relation $=$ on \mathcal{C} , thus $(G, C) = (H, D)$ means $G = H, C = D$.

1.10. Lemma. $(\mathcal{C}, \cup, \text{id}_{\mathcal{C}}), (\mathcal{C}, \cap, \text{id}_{\mathcal{C}})$ are K_4 -classes.

Proof. The identity $\text{id}_{\mathcal{C}}$ is a preorder on \mathcal{C} and the properties from 1.5. and 1.7. are simple consequences of the properties of the set-theoretical operations \cup, \cap .

1.11. Notation. Let \mathcal{C} be the same as in 1.9. We put $\times_{i \in I} (G_i, C_i) = (\times_{i \in I} G_i, \times_{i \in I} C_i)$ for any $((G_i, C_i); i \in I) \in \mathcal{S}(\mathcal{C})$; here $\times_{i \in I} G_i$ denotes the cartesian product of sets and $\times_{i \in I} C_i$ the direct (cartesian) product of ternary relations, thus $(x, y, z) \in \times_{i \in I} C_i$ for $x, y, z \in \times_{i \in I} G_i$ means $(x(i), y(i), z(i)) \in C_i$ for any $i \in I$. For $(G, C), (H, D) \in \mathcal{C}$ put $(G, C) i (H, D)$ iff there exists an isomorphism of (G, C) into (H, D) .

1.12. Lemma. (\mathcal{C}, \times, i) is a K_4 -class.

Proof is trivial. i is evidently a preorder on \mathcal{C} and \times has the properties from 1.5. and 1.7.

2. CYCLICALLY AND COCYCLICALLY ORDERED SETS

2.1. Definition. Let G be a set, C a ternary relation on G . C is called a *cyclic order* on G , iff it is:

- (i) asymmetric, i.e. $(x, y, z) \in C \Rightarrow (z, y, x) \notin C$
- (ii) transitive, i.e. $(x, y, z) \in C, (x, z, u) \in C \Rightarrow (x, y, u) \in C$
- (iii) cyclic, i.e. $(x, y, z) \in C \Rightarrow (y, z, x) \in C$.

C is called a *cocyclic order* on G , iff it is cyclic,

- (iv) reflexive, i.e. $x, y, z \in G, \text{card} \{x, y, z\} \leq 2 \Rightarrow (x, y, z) \in C$
- (v) complete, i.e. $x, y, z \in G, x \neq y \neq z \neq x \Rightarrow (x, y, z) \in C$ or $(z, y, x) \in C$

and satisfies the condition

- (vi) $x, y, z, u \in G$, pairwise distinct, $(x, y, z) \in C \Rightarrow (x, y, u) \in C$ or $(x, u, z) \in C$.

If G is a set and C a cyclic (cocyclic) order on G , then the pair (G, C) is called a *cyclically (cocyclically) ordered set*.

If C is a ternary relation on a set G , then we denote by $\text{Co}_G C$ or, briefly, $\text{Co } C$ the complement of C in G^3 , i.e. $\text{Co } C = G^3 - C$.

2.2. Lemma. Let G be a set, C a ternary relation on G . C is a cyclic order on G iff $\text{Co } C$ is a cocyclic order on G .

Proof. [7], Theorem 3.2.

2.3. Lemma. Let $(G, <)$ be an ordered set. Put for any $x, y, z \in G (x, y, z) \in C_<$ iff either $x < y < z$ or $y < z < x$ or $z < x < y$. Then $C_<$ is a cyclic order on G .

Proof. [5], Theorem 3.5.

Let $(G, <)$ be an ordered set. We call $<$ a *linear order in G* , iff there exists a subset $H \subseteq G$ such that $< \subseteq H^2$ and $<$ is a linear order on H .

2.4. Lemma. Let (G, C) be a cyclically ordered set. Then there exists a family $(<_i; i \in I)$ of linear orders in G such that $C = \bigcup_{i \in I} C_{<_i}$.

Proof. [7], Theorem 1.9. and Corollary 1.10.

2.5. Lemma. Let (G, C) be a cocyclically ordered set. Then there exists a family $(<_i; i \in I)$ of linear orders in G such that $C = \bigcap_{i \in I} \text{Co } C_{<_i}$.

Proof follows from 2.2. and 2.4.; see also [7], Theorem 3.5. and Corollary 3.6.

2.6. Definition. Let (G, C) be a cyclically ordered set. Put $w(G, C) = \min \{ \text{card } I; \text{ there exists a family } (<_i; I \in I) \text{ of orders on } G \text{ such that } C = \bigcup_{i \in I} C_{<_i} \}$, $W(G, C) = \min \{ \text{card } I; \text{ there exists a family } (<_i; i \in I) \text{ of linear orders in } G \text{ such that } C =$

$= \bigcup_{i \in I} C_{<_i}$. The cardinal $w(G, C)$ is called the *width*, the cardinal $W(G, C)$ the *strong width* of (G, C) .

If $w(G, C) = 1$, i.e. $C = C_{<}$ for a suitable order $<$ on G , then we shall say that the cyclic order C is *generated by an order*; if $W(G, C) = 1$, then C is said to be *generated by a linear order*.

2.7. Definition. Let (G, C) be a cocyclically ordered set. Put $d(G, C) = \min \{ \text{card } I; \text{ there exists a family } (<_i; i \in I) \text{ of orders on } G \text{ such that } C = \bigcap_{i \in I} \text{Co } C_{<_i} \}$, $D(G, C) = \min \{ \text{card } I; \text{ there exists a family } (<_i; i \in I) \text{ of linear orders in } G \text{ such that } C = \bigcap_{i \in I} \text{Co } C_{<_i} \}$.

If $d(G, C) = 1$, then we shall say that C is *generated by an order*; if $D(G, C) = 1$, then C is said to be *generated by a linear order*.

For the properties of characteristics $w(G, C)$, $W(G, C)$, $d(G, C)$, $D(G, C)$ see [7]. Here we recall only the following one:

2.8. Lemma. *Let (G, C) be a cocyclically ordered set. Then $d(G, C) = w(G, \text{Co } C)$, $D(G, C) = W(G, \text{Co } C)$.*

Proof. [7], Theorem 3.8.

3. DIMENSION THEORY

3.1. Notation. We denote by \mathcal{C}_y the class of all cyclically ordered sets, by $\mathcal{C}_o \mathcal{C}_y$ the class of all cocyclically ordered sets. Further, let ℓ denote the class of all cyclically ordered sets generated by an order, \mathcal{L} the class of all cyclically ordered sets generated by a linear order, $\mathcal{C}_o \ell$ the class of all cocyclically ordered sets generated by an order and $\mathcal{C}_o \mathcal{L}$ the class of all cocyclically ordered sets generated by a linear order.

3.2. Theorem. $\mathcal{C}_y(\ell; \bigcup, \text{id}_\emptyset) = \mathcal{C}_y$, $\mathcal{C}_y(\mathcal{L}; \bigcup, \text{id}_\emptyset) = \mathcal{C}_y$, $\mathcal{C}_o \mathcal{C}_y(\mathcal{C}_o \ell, \bigcap, \text{id}_\emptyset) = \mathcal{C}_o \mathcal{C}_y$, $\mathcal{C}_o \mathcal{C}_y(\mathcal{C}_o \mathcal{L}; \bigcap, \text{id}_\emptyset) = \mathcal{C}_o \mathcal{C}_y$.

Proof. Clearly, $\mathcal{C}_y(\mathcal{L}; \bigcup, \text{id}_\emptyset) \subseteq \mathcal{C}_y(\ell; \bigcup, \text{id}_\emptyset) \subseteq \mathcal{C}_y$. On the other hand, if $(G, C) \in \mathcal{C}_y$, then by 2.4. there exists a family $(<_i; i \in I)$ of linear orders in G with $C = \bigcup_{i \in I} C_{<_i}$. Thus, $(G, C) = \bigcup_{i \in I} (G, C_{<_i})$ and $(G, C_{<_i}) \in \mathcal{L}$ for any $i \in I$, i.e. $((G, C_{<_i}); i \in I) \in \mathcal{L}$. This implies $(G, C) \in \mathcal{C}_y(\mathcal{L}; \bigcup, \text{id}_\emptyset)$ and we have shown $\mathcal{C}_y \subseteq \mathcal{C}_y(\mathcal{L}; \bigcup, \text{id}_\emptyset)$. The identities $\mathcal{C}_o \mathcal{C}_y(\mathcal{C}_o \ell; \bigcap, \text{id}_\emptyset) = \mathcal{C}_o \mathcal{C}_y(\mathcal{C}_o \mathcal{L}; \bigcap, \text{id}_\emptyset) = \mathcal{C}_o \mathcal{C}_y$ follow analogously from 2.5.

3.3. Theorem. *For any $(G, C) \in \mathcal{C}_y$ it holds $(\ell; \bigcup, \text{id}_\emptyset) - \dim(G, C) = w(G, C)$, $(\mathcal{L}; \bigcup, \text{id}_\emptyset) - \dim(G, C) = W(G, C)$.*

Proof. If $w(G, C) = m$, then there exists a family $(\prec_i; i \in I)$ of orders on G with $\bigcup_{i \in I} C_{\prec_i} = C$ and $\text{card } I = m$. Then $((G, C_{\prec_i}); i \in I) \in \mathcal{S}(\ell)$ and $(G, C) = \bigcup_{i \in I} (G, C_{\prec_i})$ which implies $(\ell; \bigcup, \text{id}_{\mathcal{G}}) - \dim(G, C) \leq w(G, C)$. On the other hand, there exists a set $J \neq \emptyset$ with $\text{card } J = (\ell; \bigcup, \text{id}_{\mathcal{G}}) - \dim(G, C)$ and $(G_j, C_j) \in \ell$ for any $j \in J$ such that $(G, C) = (\bigcup_{j \in J} G_j, \bigcup_{j \in J} C_j)$. This implies $C = \bigcup_{j \in J} C_j$. Further, for any $j \in J$, there exists an order \prec_j on G_j such that $C_j = C_{\prec_j}$. Clearly, \prec_j is an order on G , and if we denote by D_j the cyclic order on G generated by \prec_j , then $D_j = C_j$ for any $j \in J$. Thus, $C = \bigcup_{j \in J} D_j$ and this implies $w(G, C) \leq (\ell; \bigcup, \text{id}_{\mathcal{G}}) - \dim(G, C)$. Analogously we can prove $(\mathcal{L}; \bigcup, \text{id}_{\mathcal{G}}) - \dim(G, C) = W(G, C)$.

3.4. Theorem. For any $(G, C) \in \mathcal{C} \circ \mathcal{C} \mathcal{Y}$ it holds $(\mathcal{C} \circ \ell; \bigcap, \text{id}_{\mathcal{G}}) - \dim(G, C) = d(G, C)$, $(\mathcal{C} \circ \mathcal{L}; \bigcap, \text{id}_{\mathcal{G}}) - \dim(G, C) = D(G, C)$.

Proof. Analogously as in the proof of 3.3. we easily see that $(\mathcal{C} \circ \ell; \bigcap, \text{id}_{\mathcal{G}}) - \dim(G, C) \leq d(G, C)$, $(\mathcal{C} \circ \mathcal{L}; \bigcap, \text{id}_{\mathcal{G}}) - \dim(G, C) \leq D(G, C)$. On the other hand, let $((G_i, C_i); i \in I) \in \mathcal{S}(\mathcal{C} \circ \ell)$ be such a family that $(G, C) = (\bigcap_{i \in I} G_i, \bigcap_{i \in I} C_i)$ and $\text{card } I = (\mathcal{C} \circ \ell; \bigcap, \text{id}_{\mathcal{G}}) - \dim(G, C)$. Then $C = \bigcap_{i \in I} C_i$; furthermore, for any $i \in I$, there exists an order \prec_i on G_i such that $C_i = \text{Co}_{G_i} C_{\prec_i}$. Put $D_i = \text{Co}_G C_{\prec_i \cap G^2}$ for any $i \in I$. Clearly, $\prec_i \cap G^2$ is an order on G , hence $C_{\prec_i \cap G^2}$ is a cyclic order on G and D_i is a cocyclic order on G generated by an order, i.e. $(G, D_i) \in \mathcal{C} \circ \ell$ for any $i \in I$. Let $x, y, z \in G$, $(x, y, z) \bar{\in} \bigcap_{i \in I} D_i$. Then there exists $i_0 \in I$ such that $(x, y, z) \bar{\in} D_{i_0}$, i.e. $(x, y, z) \in C_{\prec_{i_0} \cap G^2}$. This implies $x <_{i_0} y$, $y <_{i_0} z$ or $y <_{i_0} z$, $z <_{i_0} x$ or $z <_{i_0} x$, $x <_{i_0} y$. Thus $(x, y, z) \in C_{\prec_{i_0}}$ and, hence, $(x, y, z) \bar{\in} \bigcap_{i \in I} \text{Co}_{G_i} C_{\prec_i}$. We have proved $\bigcap_{i \in I} \text{Co}_{G_i} C_{\prec_i} \subseteq \bigcap_{i \in I} D_i$. Suppose that $x, y, z \in G$, $(x, y, z) \bar{\in} \bigcap_{i \in I} \text{Co}_{G_i} C_{\prec_i}$. Then $(x, y, z) \in G^3$ and there exists $i_0 \in I$ such that $(x, y, z) \in C_{\prec_{i_0}}$. Thus either $x <_{i_0} y$, $y <_{i_0} z$ or $y <_{i_0} z$, $z <_{i_0} x$ or $z <_{i_0} x$, $x <_{i_0} y$. As $x, y, z \in G$, we have $(x, y, z) \in C_{\prec_{i_0} \cap G^2}$ so that $(x, y, z) \bar{\in} \bigcap_{i \in I} \text{Co}_G C_{\prec_i \cap G^2} = \bigcap_{i \in I} D_i$ and we have proved $\bigcap_{i \in I} D_i \subseteq \bigcap_{i \in I} \text{Co}_{G_i} C_{\prec_i}$. Thus $\bigcap_{i \in I} D_i = \bigcap_{i \in I} \text{Co}_{G_i} C_{\prec_i}$, which implies $(G, C) = \bigcap_{i \in I} (G, D_i)$ where D_i is a cocyclic order on G generated by an order on G . Hence $d(G, C) \leq (\mathcal{C} \circ \ell; \bigcap, \text{id}_{\mathcal{G}}) - \dim(G, C)$. For the second assertion the proof is similar.

3.5. Theorem. $\mathcal{C} \circ \mathcal{C} \mathcal{Y}(\mathcal{C} \circ \ell; \mathbf{X}, i) = \mathcal{C} \circ \mathcal{C} \mathcal{Y}$, $\mathcal{C} \circ \mathcal{C} \mathcal{Y}(\mathcal{C} \circ \mathcal{L}; \mathbf{X}, i) = \mathcal{C} \circ \mathcal{C} \mathcal{Y}$.

Proof. It suffices to show $\mathcal{C} \circ \mathcal{C} \mathcal{Y} \subseteq \mathcal{C} \circ \mathcal{C} \mathcal{Y}(\mathcal{C} \circ \mathcal{L}; \mathbf{X}, i)$, because the inclusion $\mathcal{C} \circ \mathcal{C} \mathcal{Y}(\mathcal{C} \circ \mathcal{L}; \mathbf{X}, i) \subseteq \mathcal{C} \circ \mathcal{C} \mathcal{Y}(\mathcal{C} \circ \ell; \mathbf{X}, i) \subseteq \mathcal{C} \circ \mathcal{C} \mathcal{Y}$ is trivial. Let $(G, C) \in \mathcal{C} \circ \mathcal{C} \mathcal{Y}$. By 2.5. there exists a family $(\prec_i; i \in I)$ of linear orders in G such that $C = \bigcap_{i \in I} \text{Co } C_{\prec_i}$. For any $i \in I$, put $G_i = G$, $C_i = \text{Co}_G C_{\prec_i}$. Then $((G_i, C_i); i \in I) \in \mathcal{S}(\mathcal{C} \circ \mathcal{L})$. Further-

more, $\prod_{i \in I} (G_i, C_i) = (\prod_{i \in I} G_i, \prod_{i \in I} \text{Co}_G C_{<_i})$. For any $x \in G$ and any $i \in I$, put $f(x)(i) = x$. Thus, f is a mapping of G into $\prod_{i \in I} G_i$; clearly, f is an injection. Let $x, y, z \in G$, $(x, y, z) \in C$. Then $(x, y, z) \in \text{Co}_G C_{<_i}$ for any $i \in I$, i.e. $(f(x)(i), f(y)(i), f(z)(i)) \in C_i$ for any $i \in I$. Hence $(f(x), f(y), f(z)) \in \prod_{i \in I} C_i$. If $(f(x), f(y), f(z)) \in \prod_{i \in I} C_i$, then we have $(x, y, z) = (f(x)(i), f(y)(i), f(z)(i)) \in C_i$ for any $i \in I$, i.e. $(x, y, z) \in \bigcap_{i \in I} \text{Co}_G C_{<_i} = C$. We have proved that f is an isomorphism of (G, C) into $\prod_{i \in I} (G_i, C_i)$ so that $(G, C) \in \mathcal{X}(\prod_{i \in I} (G_i, C_i))$ and $(G, C) \in \mathcal{C}o\mathcal{C}y(\mathcal{C}o\mathcal{L}; \mathbf{X}, i)$. Thus, $\mathcal{C}o\mathcal{C}y \subseteq \mathcal{C}o\mathcal{C}y(\mathcal{C}o\mathcal{L}; \mathbf{X}, i)$.

3.6. Theorem. For any $(G, C) \in \mathcal{C}o\mathcal{C}y$ it holds $(\mathcal{C}o\mathcal{L}; \mathbf{X}, i) - \dim(G, C) = d(G, C)$, $(\mathcal{C}o\mathcal{L}; \mathbf{X}, i) - \dim(G, C) = D(G, C)$.

Proof. From the proof of 3.5. it follows that $(\mathcal{C}o\mathcal{L}; \mathbf{X}, i) - \dim(G, C) \leq d(G, C)$, $(\mathcal{C}o\mathcal{L}; \mathbf{X}, i) - \dim(G, C) \leq D(G, C)$. On the other hand, let $\{(G_i, C_i); i \in I\} \in \mathcal{S}(\mathcal{C}o\mathcal{L})$ be such a family that $\text{card } I = (\mathcal{C}o\mathcal{L}; \mathbf{X}, i) - \dim(G, C)$ and there is an isomorphism f of (G, C) into $(\prod_{i \in I} G_i, \prod_{i \in I} C_i)$. For any $i \in I$, there exists an order $<_i$ on G_i such that $C_i = \text{Co}_{G_i} C_{<_i}$. For $i \in I$ and $x, y \in G$, put $x <_i y$ iff $f(x)(i) <_i f(y)(i)$. We show that $<_i$ is an order on G . Indeed, if $x \in G$ is arbitrary, then $x <_i x$ is equivalent to $f(x)(i) <_i f(x)(i)$ which never holds. Hence, $<_i$ is irreflexive. If $x, y, z \in G$ and $x <_i y$, $y <_i z$, then $f(x)(i) <_i f(y)(i)$, $f(y)(i) <_i f(z)(i)$ which implies $f(x)(i) <_i f(z)(i)$, thus $x <_i z$. Hence, $<_i$ is transitive. Further, we prove $C = \bigcap_{i \in I} \text{Co}_G C_{<_i}$. Indeed, suppose $x, y, z \in G$. If $(x, y, z) \in C$, then $(f(x)(i), f(y)(i), f(z)(i)) \in \text{Co}_{G_i} C_{<_i}$ for any $i \in I$. Suppose that there exists $i_0 \in I$ such that $(x, y, z) \notin \text{Co}_G C_{<_{i_0}}$. Thus, $(x, y, z) \in C_{<_{i_0}}$ which means either $x <_{i_0} y$, $y <_{i_0} z$ or $y <_{i_0} z$, $z <_{i_0} x$ or $z <_{i_0} x$, $x <_{i_0} y$. By definition of $<_{i_0}$, this means either $f(x)(i_0) <_{i_0} f(y)(i_0)$, $f(y)(i_0) <_{i_0} f(z)(i_0)$ or $f(y)(i_0) <_{i_0} f(z)(i_0)$, $f(z)(i_0) <_{i_0} f(x)(i_0)$ or $f(z)(i_0) <_{i_0} f(x)(i_0)$, $f(x)(i_0) <_{i_0} f(y)(i_0)$. Hence, $(f(x)(i_0), f(y)(i_0), f(z)(i_0)) \in C_{<_{i_0}}$ which is a contradiction for $(f(x)(i_0), f(y)(i_0), f(z)(i_0)) \in \text{Co}_{G_{i_0}} C_{<_{i_0}}$. Thus, $C \subseteq \bigcap_{i \in I} \text{Co}_G C_{<_i}$. Assume that there exists $(x, y, z) \in \bigcap_{i \in I} \text{Co}_G C_{<_i} - C$. Then $(f(x), f(y), f(z)) \notin \prod_{i \in I} \text{Co}_{G_i} C_{<_i}$; hence, there exists $i_0 \in I$ such that $(f(x)(i_0), f(y)(i_0), f(z)(i_0)) \in C_{<_{i_0}}$. This means either $f(x)(i_0) <_{i_0} f(y)(i_0)$, $f(y)(i_0) <_{i_0} f(z)(i_0)$ or $f(y)(i_0) <_{i_0} f(z)(i_0)$, $f(z)(i_0) <_{i_0} f(x)(i_0)$ or $f(z)(i_0) <_{i_0} f(x)(i_0)$, $f(x)(i_0) <_{i_0} f(y)(i_0)$. By definition of $<_{i_0}$, we obtain either $x <_{i_0} y$, $y <_{i_0} z$ or $y <_{i_0} z$, $z <_{i_0} x$ or $z <_{i_0} x$, $x <_{i_0} y$, i.e. $(x, y, z) \in C_{<_{i_0}}$ which contradicts our hypothesis. Thus, $\bigcap_{i \in I} \text{Co}_G C_{<_i} = C$ and we have proved $d(G, C) \leq \text{card } I = (\mathcal{C}o\mathcal{L}; \mathbf{X}, i) - \dim(G, C)$. If each $<_i$ is a linear order in G_i , then $<_i$ is a linear order in G which proves the second assertion of the theorem.

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