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AN OSCILLATION CRITERION FOR nTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENTS

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Recently Ohriska [3] established an oscillation criterion for the Emden Fowler equation

$$x^{(n)} + q(t)\,|x[g(t)]|^{\alpha}\operatorname{sgn} x[g(t)] = 0\,,\quad \alpha > 0\,,\quad \left({}' = \frac{\mathrm{d}}{\mathrm{d}x}\right).$$

The purpose of this note is to establish a similar criterion for the n-th order functional equation for n even

$$x^{(n)} + p(t)\,|x^{(n-1)}|^{\beta}\,x^{(n-1)} + q(t)\,f(x[g(t)]) = 0\,,\quad \beta \geqq 0\,, \tag{1}$$

where $p, q, g\colon [t_0, \infty) \to [0, \infty)$, $f\colon R \to R$ are continuous, $xf(x) > 0$ for $x \neq 0$, $g(t) \leqq t$ and $\lim_{t\to\infty} g(t) = \infty$.

Let $E_{t_0} = \{s \mid s = g(t) \leqq t_0 \text{ for } t \geqq t_0\} \cup \{t_0\}$. By a solution of (1) at t_0 is meant a function $x\colon E_{t_0} \cup [t_0, t_1) \to R$ for some $t_1 > t_0$, which satisfies (1) for all $t \in [t_0, t_1)$. All solutions of (1) defined at t_0 are assumed to be continuable to infinity for every $t_0 \geqq 0$. A solution $x(t)$ of (1) is said to be *oscillatory* if $x(t)$ has zero for arbitrarily large t. Equation (1) is said to be *oscillatory* if every solution of (1) is oscillatory.

We will have an occasion to use the following two Lemmas given in [2].

Lemma 1. *Let u be a positive and n-times differentiable function on $[t_0, \infty)$. If $u^{(n)}(t)$ is of constant sign and not identically zero in any interval of the form $[t_1, \infty)$, there exists a $t_u \geqq t_0$ and an integer l, $0 \leqq l \leqq n$ with $n + l$ even for $u^{(n)}(t) \geqq 0$ or $n + l$ odd for $u^{(n)}(t) \leqq 0$ and such that $l > 0$ implies that $u^{(k)}(t) > 0$ for $t \geqq t_u$, $(k = 0, 1, \ldots, l - 1)$ and $l \leqq n - 1$ implies that $(-1)^{l+k}\,u^{(k)}(t) > 0$ for $t \geqq t_u$, $(k = l, l + 1, \ldots, n - 1)$.*

Lemma 2. *If the function u is as in Lemma 1 and*

$$u^{(n-1)}(t)\,u^{(n)}(t) \leqq 0 \quad \text{for} \quad t \geqq t_u\,,$$

then for every λ, $0 < \lambda < 1$, there exists $M > 0$ such that

$$u(\lambda t) \geqq M t^{n-1} |u^{(n-1)}(t)| \quad \textit{for all large } t \, .$$

Also we need the following Lemma:

Lemma 3. *Let*

$$\left(1 + \int_T^t p(s)\, ds\right)^{-1/\beta} \notin \mathscr{L}(T, \infty) \quad \textit{if} \quad \beta > 0 \tag{2}$$

and

$$\int_T^\infty \exp\left(-\int_T^s p(\tau)\, d\tau\right) ds = \infty \quad \textit{if} \quad \beta = 0 \, .$$

Then if x is a nonoscillatory solution of (1), *we must have $x(t)\, x^{(n-1)}(t) > 0$ for all large t.*

Proof. The proof is similar to that of Lemma 3 in [1] and hence is omitted.

We let

$$\gamma(t) = \sup \{s \geqq t_0 \mid g(s) \leqq t\} \quad \text{for} \quad t \geqq t_0 \, .$$

Theorem 1. *In addition to condition* (2) *we assume*

$$f'(x) \geqq 0 \quad \text{for} \quad x \neq 0 \, , \quad \left(' = \frac{d}{dx}\right), \tag{3}$$

and

$$\lim_{x \to \infty} \frac{x}{f(x)} = 0 \, . \tag{4}$$

If (1) *has an unbounded nonoscillatory solution, then*

$$\limsup_{t \to \infty} t^{n-1} \int_{\gamma(t)}^\infty q(s)\, ds = 0 \, . \tag{5}$$

Proof. Let $x(t)$ be a nonoscillatory unbounded solution of (1). Without loss of generality we assume that $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqq t_1 \geqq t_0$. Now, by Lemma 3, there exists a $t_2 \geqq t_1$ such that $x^{(n-1)}(t) > 0$ for $t \geqq t_2$. Equation (1), then becomes

$$x^{(n)} + q(t) f(x[g(t)]) \leqq 0 \, . \tag{6}$$

By Lemma 1, there exists a $t_3 \geqq t_2$ such that $\dot{x}(t) > 0$ for $t \geqq t_3$. Using Lemma 2, we can find constants $M > 0$ and $t_4 \geqq t_3$ such that

$$x[\lambda t] \geqq M t^{n-1} x^{(n-1)}(t) \quad \text{for} \quad t \geqq t_4 \quad \text{and some} \quad \lambda \in (0, 1) \, .$$

From (6) and the fact that $x^{(n-1)}(t)$ is non-increasing we have

$$x^{(n-1)}(t) \geqq \int_t^\infty q(s) f(x[g(s)])\, ds \, .$$

Thus

$$x(t) \geqq x[\lambda t] \geqq Mt^{n-1} x^{(n-1)}(t) \geqq Mt^{n-1} \int_t^\infty q(s) f(x[g(s)])\, \mathrm{d}s\,, \quad \text{for} \quad t \geqq t_4\,.$$

We know that $\gamma(t) \geqq t$ and $g(s) \geqq t$ if $s \geqq \gamma(t)$. Since $\dot{x}(t) > 0$ we have

(7) $$x[g(s)] \geqq x(t) \quad \text{for} \quad s \geqq \gamma(t) \quad \text{and} \quad x(t) \geqq Mt^{n-1} f(x(t)) \int_{\gamma(t)}^\infty q(s)\, \mathrm{d}s\,.$$

Now from (4) and (7) we have

$$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^\infty q(s)\, \mathrm{d}s = 0\,.$$

The proof is complete.

Theorem 2. *Let conditions* (2) *and* (3) *hold, and*

(8) $$\limsup_{t\to\infty} t^{n-1} \int_t^\infty q(s)\, \mathrm{d}s = \infty\,.$$

Then all nonoscillatory solutions of (1) *are unbounded.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1) such that $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqq t_1 \geqq t_0$. As in the proof of Theorem 1 we get

$$x(t) \geqq Mt^{n-1} \int_t^\infty q(s) f(x(g(s)))\, \mathrm{d}s\,.$$

Since $s \geqq t \geqq t_4 \geqq t_1 = \gamma(t_0)$ and $x(t)$ is non-decreasing we have

$$x(t) \geqq Mt^{n-1} f(x(t_4)) \int_t^\infty q(s)\, \mathrm{d}s\,.$$

From which, in view of (8) it follows that

$$\lim_{t\to\infty} x(t) = \infty\,.$$

This completes the proof of the theorem.

Theorem 3. *Let conditions* (2), (3), (4) *and* (8) *hold, and*

(9) $$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^\infty q(s)\, \mathrm{d}s > 0\,,$$

then any solution of (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1) such that $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqq t_1 \geqq t_0$. By condition (9) and Theorem 1, $x(t)$ must be bounded. On the other hand by condition (8) and Theorem 2, $x(t)$ must be unbounded. This is a contradiction and the theorem is proved.

Remarks. 1. If $p(t) = 0$, $f(x) = |x|^\alpha \operatorname{sgn} x$, $\alpha > 0$, the Ohriska's results and our results are the same. Moreover our proofs are short and simple.

2. Condition (8) in Theorem 2 is only a sufficient condition. For illustration consider the equation

$$\ddot{x} + \frac{1}{4t^2} x = 0 .$$

Now

$$\limsup_{t \to \infty} t \int_t^\infty \frac{1}{4s^2} \, ds = \frac{1}{4} \neq \infty .$$

The above equation has the nonoscillatory unbounded solutions $x_1(t) = \sqrt{t}$ and $x_2(t) = (\sqrt{t}) \ln t$.

For illustration we consider the equation

$$x^{(n)} + t^l |x^{(n-1)}|^m x^{(n-1)} + \frac{1}{t^{n-1}} f(x[g(t)]) = 0 ,$$

for n even, $t \geqq T$ for suitable constant T, $l + 1 \leqq m$. We let

$$g(t) = ct \quad \text{or} \quad t^c \quad \text{for} \quad c \in (0, 1] \quad \text{and} \quad t \geqq 1$$

and

$$f(x) = x^{\alpha_1} \quad \text{or} \quad x^\alpha e^{x^2} \quad \text{or} \quad x^\alpha \log(e + x^2) \quad \text{or} \ \sinh x ,$$

where α, α_1 are the ratio of two positive odd integers and $\alpha_1 > 1$. The above equation is oscillatory by our Theorem 3, while Theorem 3 in [3] fails to apply.

We believe that the above conclusion does not appear to be deducible from other known oscillation criteria.

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