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SEMIGROUPS AND RINGS WHOSE PROPER ONE-SIDED IDEALS ARE POWER JOINED

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The main result of the present paper consists in a characterization of semigroups whose proper one-sided ideals are power joined (Th. 1.6). A theorem of Pondělíček on uniform semigroups, contained in [3], is found again as a corollary of this result. Moreover, with regard to the fact that every one-sided ideal is a biideal (biideal of a semigroup $S$ is a subsemigroup $B$ of $S$ such that $BSB \subseteq B$), Th. 1.6 provides also an answer to the question put in the Mathematical Reviews (82g: 20097) by the reviewer of the note [3]. The second section of the work contains a characterization of rings whose proper one-sided ideals are multiplicatively power joined semigroups.

1. We start by proving some lemmas which will enable us to establish the main theorem. We remember that by a power joined semigroup we mean a semigroup $S$ such that $a^h = b^k$ for every $a, b \in S$ and $h, k$ positive integers (see [4], II.7.7).

**Lemma 1.1.** A semigroup $S$ whose proper left (right) ideals of the form $Sb$ ($bS$) ($b \in S$) are power joined is a semilattice of archimedean semigroups.

**Proof.** Let $a, b \in S$ with $a = xby$ ($x, y \in S^1$). Then $a^2 = (xbyx)by$ with $z = xbyx \in S$. If $Sb = S$, it follows that $z = wb$ for some $w \in S$, and therefore $a^2 = zby = wb^2y$. If on the contrary, $Sb \subset S$, since $Sb$ is power joined, there are two positive integers $h, k$ such that $(yzb)^h = b^{2k}$, and it results that $a^{2h+1} = (zby)^{h+1} = zb(yzb)^h y = zb^{2k+1}y$. Thus in any case $b^2$ divides a power of $a$, which suffices to conclude that $S$ is a semilattice of archimedean semigroups (see [5], Th. 2.1).

**Lemma 1.2.** A non archimedean semigroup $S$ whose proper one-sided ideals are power joined is a semilattice of two semigroups $M$ and $S \setminus M$, where $M$ is power joined and coincides with the greatest ideal of $S$, and $S \setminus M$ is a group. Moreover, $S$ has an identity, which is the identity of $S \setminus M$.

**Proof.** By Lemma 1.1, $S$ is a semilattice of archimedean semigroups, so it is Putcha's $Q$-semigroup (see [1], Definitions 1.1 and 1.4). Therefore it follows from Corollary 1.5 of [1] that $S$ is a semilattice of two semigroups: $M$ which is power
joined and the greatest ideal of $S$, and $S \setminus M$. It remains to prove that $S \setminus M$ is a group, whose identity is the identity of $S$. In fact, if $L$ is a proper left ideal of $S \setminus M$, it is immediate to verify that $L \cup M$ is a proper left ideal of $S$, and therefore a power joined semigroup. But this is a contradiction, since power joined semigroups cannot be disjoint unions of proper subsemigroups. Hence $S \setminus M$ is left simple. In the same way we find that $S \setminus M$ is right simple. Thus $S \setminus M$ is a group. Now, if $u$ is the identity of $S \setminus M$, $Su \subset S$ implies that $Su$ is power joined. Then, for every $x \in M$, there exists a positive integer $m$ such that $(xu)^m = u$, but this implies $u \in M$, a contradiction. Thus $Su = S$, and analogously $uS = S$, which means that $u$ is the identity of $S$.

Lemma 1.3. A non simple archimedean semigroup whose proper one-sided ideals are power joined is power joined.

Proof is immediate.

Lemma 1.4. A simple semigroup $S$ whose proper one-sided ideals are power-joined has at least an idempotent.

Proof. The lemma is obvious if $S$ is a group. Otherwise, we may suppose that $S$ is not left simple. Therefore, there exists $a \in S$ such that $Sa \subset S$. Moreover, $S$ being simple, we have $Sa^2 S = S$, hence $a = xa^2 y$ for some $x, y \in S$. Since $Sa$ is power joined, there are two positive integers $h, k$ such that $(xa)^h a = a^{2k}$, whence $a = (xa) ay = (xa)^h ay^h = a^{2k+1} y^h$. Hence $a^2 = a^{2k-1} a^2 y^h a$ with $y^h a \in Sa$. Then there exist two positive integers $m, n$ such that $(y^h a)^m = a^{2n}$, and consequently, $a^2 = (a^{2k-1})^m a^2 (y^h a)^m = a^{(2k-1)m+2+2n}$. So $S$ has an idempotent.

Lemma 1.5. A simple semigroup $S$ whose proper one-sided ideals are power-joined is either a group or a left (right) zero-semigroup of two periodic groups.

Proof. Since a simple semigroup with a unique idempotent is a group, we may suppose that $S$ contains two idempotents $e, f$ with $e \neq f$ (Lemma 1.4). First we remark that $Se \subset S$ implies $e = (fe)^h$ for some positive integer $h$, since $Se$ is a power joined semigroup containing $e$ and $fe$. Hence $e = fe$. Analogously, $fS \subset S$ implies $f = fe$. Since $e \neq f$, we have either $Se = S$ or $fS = S$. In the same way we find that necessarily either $eS = S$ or $fS = S$.

If $Se = eS = S$ or $fS = fS = S$, the semigroup $S$ has an identity. If, on the contrary, $Se = fS = S$ ($eS = fS = S$), $S$ has two right (left) identities. It is immediate to verify that in the first case $S$ cannot have a third idempotent different from $e$ and $f$. In the second case, every other idempotent is a right (left) identity. In both cases one of the idempotents of $S$ is primitive (since their number is finite when $S$ has an identity); since they are not comparable in the other case), so $S$ is completely simple, i.e. $S$ is a rectangular band of groups. This leads to a contradiction when $S$

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1) $S$, being simple with order greater than $1$, cannot have a zero.
has an identity, so all idempotents of \( S \) are right (left) identities. Now, let \( G_x \) and \( G_y \) be two maximal subgroups of \( S \) with \( G_x \neq G_y \). From the fact that the idempotents of \( S \) are right (left) identities it follows that \( S \) is a left (right) zero-semigroup of groups, whence \( (G_x \cup G_y)S \leq G_x \cup G_y \). Then, if \( G_x \cup G_y \subseteq S \), \( G_x \cup G_y \) has to be power joined, a contradiction. Thus \( G_x \cup G_y = S \) and \( S \) is a left (right) zero-semigroup of two groups \( G_x, G_y \). Finally, \( G_x \) and \( G_y \) are periodic since they are proper right (left) ideals of \( S \) and consequently, power joined groups.

Now we are able to state the following result:

**Theorem 1.6.** Let \( S \) be a semigroup whose proper one-sided ideals are power joined. Then \( S \) satisfies one of the following conditions:

i) \( S \) is power joined,

ii) \( S \) is a group,

iii) \( S \) is a left (right) zero-semigroup of two periodic groups,

iv) \( S \) is a semilattice of two semigroups \( M \) and \( S \setminus M \), where \( M \) is power joined and coincides with the greatest ideal of \( S \), and \( S \setminus M \) is a group. Moreover, the identity of \( S \setminus M \) is the identity for \( S \).

Conversely, if \( S \) is a semigroup of type i), ii), iii) or iv), every proper one-sided ideal of \( S \) is power joined.

**Proof.** The first part of the statement follows from Lemmas 1.2, 1.3 and 1.5. The converse is immediate.

We recall that a semigroup \( S \) is said to be uniform if every two left ideals of \( S \) and every two right ideals of \( S \) have a non-empty intersection (see [3], p. 331). According to this definition, we may find again Th. 1 of [3] as a corollary of the above Th. 1.6. In fact, when \( S \) is uniform, the case iii) of the statement of Th. 1.6 can not occur, since the components of a left (right) zero-semigroup of groups are disjoint right (left) ideals, and the converse is obvious.

**Addendum.** From the above Th. 1.6 and from Th. 4 of [7] the following Theorem can be immediately deduced: “\( S \) is a semigroup whose proper subsemigroups are power joined if and only if \( S \) is either power joined or a band of order two”. This result extends Th. 2 of [8], which has come to the authors’ knowledge when the manuscript was already sent to the Editor.

2. In this section we shall prove a theorem for rings analogous to Th. 1.6. In the sequel we shall denote by \((R, \cdot)\) the multiplicative semigroup of a ring \( R \).

**Theorem 2.1.** Let \( R \) be a ring whose proper one-sided ideals are multiplicatively power joined semigroups. Then \( R \) satisfies one of the following conditions:

i) \( R \) is a nilring,
ii) $R$ is a ring with identity and $(R, \cdot)$ is a semilattice of two semigroups $M$ and $R \setminus M$, where $M$ is a nilring and coincides with the greatest ideal of $R$, and $R \setminus M$ is a group.

Conversely, if $R$ is a ring of type i) or ii), every proper one-sided ideal of $R$ is a multiplicatively power joined semigroup.

Proof. Let $R$ be a ring whose proper one-sided ideals are multiplicatively power joined semigroups. If $(R, \cdot)$ is archimedean, it has to be a nilsemigroup, since it contains the zero, and in this case $R$ is a nilring. Then, let us suppose that $(R, \cdot)$ is not archimedean. We note that every left ideal of $(R, \cdot)$ of the form $Rb$ ($b \in R$) is an additive subgroup of $R$, hence it is a left ideal of $R$. Analogously, every right ideal of $(R, \cdot)$ of the form $aR$ ($a \in R$) is a right ideal of $R$. Therefore, by Lemma 1.1, $(R, \cdot)$ is a semilattice of archimedean semigroups. Since $(R, \cdot)$ is not archimedean, it contains a proper prime ideal $M$ (see [1], Th. 1.3). For every $a, b \in M$, we have $aR \subseteq M$ and $Rb \subseteq M$; hence $aR$ and $Rb$ are proper one-sided ideals of $R$, so they are multiplicatively power joined semigroups. Since $ab \in aR \cap Rb$, there are three positive integers $h, k, l$ such that $a^{2h} = (ab)^k = b^{2l}$. Thus $M$ is power joined and, containing zero, it has to be a nilsemigroup. Then, let $a \in M$, $c \in R \setminus M$. If $Rc \subset R$, since $Rc$ and $aR$ are power joined and $ac \in aR \cap Rc$, we may conclude as above that $a^{2h} = c^{2l}$ for some positive integers $h, l$. This is a contradiction, since $a$ and $c$ are in disjoint semigroups. Thus, for every $c \in R \setminus M$ we have $Rc = R$. Analogously we find $cR = R$. This implies that for every $c, d \in R \setminus M$ there exist $x, y \in R$ such that $d = xc = cy$. Since $x \in M$ ($y \in M$) implies $d \in M$, a contradiction, we have $x, y \in R \setminus M$, which enables us to conclude that $R \setminus M$ is a group. If $u$ is the identity of $R \setminus M$, the relations $Ru = uR = R$ imply that $u$ is the identity of $R$. Thus it remains to prove that $M$ is the greatest ideal of $R$. In fact, since $(R, \cdot)$ is a semilattice of archimedean semigroups, such that every element has a power in a subgroup, the set of all nilpotents of $R$ is an ideal of $R$ (see [6], Th. 8), which obviously coincides with $M$. The maximality of $M$ is guaranteed by the fact that $R \setminus M$ is a group.

The converse is immediate.

Remark 2.2. The class of rings of type ii) in the statement of Th. 2.1 contains all division rings; nevertheless it may be interesting to note that this class is wider than that of division rings. This is proved by the following example. Let $R$ be the set of square matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + h \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$$

($a, b$ real numbers; $h$ complex; $i = \sqrt{-1}$). It is a routine verification to prove that $R$ is a ring with identity with respect to the usual sum and product of matrices. Moreover, it is immediate to show that the subset $M$ of $R$ containing the matrices

2) Here and in [1] "prime ideal" means, following Clifford, an ideal $I$ of a semigroup $S$ such that $S \setminus I$ is a subsemigroup. Such ideals are called "completely prime" by Petrich.

124
is a nilring of order greater than 1 and it is the greatest ideal of \( R \), while the subset \( R \setminus M \) is a multiplicative group. So \( R \) is a ring of type ii) and it is not a division ring.

References


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