Czechoslovak Mathematical Journal

Fabio Zanolin

Example of a convergence commutative group which is not separated

Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 2, 169–171

Persistent URL: http://dml.cz/dmlcz/101939

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of the Czechoslovak Academy of Sciences
V. *34* (*109*), PRAHA 24. 6. 1984, No 2

EXAMPLE OF A CONVERGENCE COMMUTATIVE GROUP WHICH IS NOT SEPARATED

FABIO ZANOLIN, Trieste

(Received January 25, 1977)

1. INTRODUCTION

It is well-known that a sequential convergence space with unique sequential limits need not be separated (cf. [1]). J. Novák at the Kanpur Topological Conference asked whether each sequential convergence group (with unique sequential limits) is separated (Problem 12 in [2]). In [3] the following construction of sequential convergence groups (not necessarily with unique sequential limits) has been developed. Starting with a set A, the free Z-module G generated by A can be equipped with the smallest multivalued sequential convergence of the type L^*, compatible with the group structure of G, in which a given set of sequences of points of G converges to the neutral element 0 of G. The fact that G is a free Z-module guarantees that the resulting convergence group has some nice properties. Using the same type of construction, in the present paper we give a negative answer to the question asked by J. Novák.

2. PRELIMINARIES

In this section we recall some facts about sequential convergence groups (see e.g. [2]) and the free Z-module technique from [3].

Throughout the paper Z denotes the group of integers, N the set of natural numbers (i.e. positive integers), N^N the set of all mappings of N into N and $\mathcal{S}$ the set of all increasing mappings in N^N. Let X be an infinite set. If $S = (x_n)$ is a sequence in X (i.e. a mapping of N into X the n-th term of which is $S(n) = x_n$) and $s \in \mathcal{S}$, then $S \circ s$ denotes the sequence in X the n-th term of which is $(S \circ s)(n) = x_{s(n)}$. For $x \in X$ the symbol (x) denotes the constant sequence generated by x (i.e. $(x)(n) = x$ for all $n \in N$) and $\{x\}$ denotes the subset of X the only element of which is x.

Let G be a commutative group. For $S, T \in G^N$ define $(S + T)(n) = S(n) + T(n)$ and $(-T)(n) = -T(n)$, $n \in N$. Then G^N is a commutative group. Let $\mathfrak{G}$ be a subset of $G^N \times G$ satisfying axioms

($\mathscr{L}_0$) If $(S, x) \in \mathfrak{G}$ and $(S, y) \in \mathfrak{G}$, then $x = y$;

($\mathscr{L}_1$) $((x), x) \in \mathfrak{G}$ for each $x \in G$;

($\mathscr{L}_2$) If $(S, x) \in \mathfrak{G}$, then $(S \circ s, x) \in \mathfrak{G}$ for each $s \in \mathscr{S}$;

($\mathscr{L}_3$) $(S, x) \in \mathfrak{G}$ whenever for each $s \in \mathscr{S}$ there exists $t \in \mathscr{S}$ such that $(S \circ s \circ t, x) \in \mathfrak{G}$;

($\mathscr{S}^*\mathscr{G}$) If $(S, x) \in \mathfrak{G}$ and $(T, y) \in \mathfrak{G}$, then $(S - T, x - y) \in \mathfrak{G}$.

If $(S, x) \in \mathfrak{G}$, then we say that the sequence S $\mathfrak{G}$-converges to x. For $A \subset G$ define $\gamma A = \{x \in G;\ (S, x) \in \mathfrak{G} \text{ for some } S \in A^N\}$. Then G equipped with $\mathfrak{G}$ and γ is said to be a convergence commutative group (cf. [2]).

Let G be a commutative group and let B be a subset of G^N. Let δB be the set of all sequences in G of the form $S \circ s$ with $S \in B$ and $s \in \mathscr{S}$, let $\langle \delta B \rangle$ be the smallest subgroup of G^N containing δB, and let $\zeta\langle \delta B \rangle$ be the set of all sequences S in G such that for each $s \in \mathscr{S}$ there exists $t \in \mathscr{S}$ such that $S \circ s \circ t \in \langle \delta B \rangle$. Define $\mathfrak{G} \subset G^N \times G$ as follows: $(S, x) \in \mathfrak{G}$ whenever $S - (x) \in \zeta\langle \delta B \rangle$. By Corollary in [3], $\mathfrak{G}$ satisfies axioms ($\mathscr{L}_1$), ($\mathscr{L}_2$), ($\mathscr{L}_3$) and ($\mathscr{S}^*\mathscr{G}$). Further, by Lemma 2 in [3], $\mathfrak{G}$ satisfies ($\mathscr{L}_0$) iff (0) is the only constant sequence in G belonging to $\zeta\langle \delta B \rangle$.

Let A be an infinite set and let G be the free Z-module generated by A. Then G is equipped with a commutative group structure. Recall that elements of G can be represented by reduced linear combinations $\sum_{k=1}^{h} z_k a_k$, where h is a nonnegative integer, $z_k \in Z \setminus \{0\}$, $a_k \in A$ and $a_k \neq a_l$ whenever $k \neq l$. For $x \in G$, $x = \sum_{k=1}^{h} z_k a_k$, define $\operatorname{gen}(x) = \{a_k;\ k = 1, \ldots, h\}$. Note that for $h = 0$ we have $\operatorname{gen}(x) = \emptyset$ and x is the neutral element 0 of G. Also, two elements $\sum_{k=1}^{h} z_k a_k$ and $\sum_{k=1}^{g} w_k b_k$ of G are equal iff $h = g$ and there is a permutation p of the set $\{1, \ldots, h\}$ such that $a_k = b_{p(k)}$ and $z_k = w_{p(k)}$ for all $k \in \{1, \ldots, h\}$.

3. THE EXAMPLE

We start with the following well-known example of a Fréchet space X (i.e. X is a topological space such that whenever a point x belongs to a closure of a set A, then there is a sequence in A converging in X to x) which has unique sequential limits but fails to be Hausdorff. The space X consists of a double sequence $Y = \{a(i, j);\ i, j = 1, 2, \ldots\}$ and two other distinct points a, b. Points $a(i, j)$ are isolated. A neighbourhood base at a is formed by sets $\{a\} \cup A(f)$, where f is a mapping of N into N and $A(f) = \{a(i, j) \in Y;\ j > f(i)\}$. A neighbourhood base at b is formed by sets $\{b\} \cup A(k)$, where $k \in N$ and $A(k) = \{a(i, j) \in Y;\ i > k\}$. Note that for each fixed $k \in N$ the sequence $U_k \in Y^N$ defined by $U_k(n) = a(k, n)$ converges in X to a, and for each mapping $f \in N^N$ the sequence $V_f \in Y^N$ defined by $V_f(n) = a(n, f(n))$ converges in X to b.

Now, consider the subset $A = \{a\} \cup Y$ of X. Let G be the free Z-module generated by A. We are going to equip G with a sequential convergence $\mathfrak{G} \subset G^N \times G$ satisfying axioms $(\mathscr{L}_0)$, $(\mathscr{L}_1)$, $(\mathscr{L}_2)$, $(\mathscr{L}_3)$ and $(\mathscr{S}^*\mathscr{G})$ such that the following condition

$(*)$ $(U_k, a) \in \mathfrak{G}$ for each $k \in N$ and $(V_f, 0) \in \mathfrak{G}$ for each $f \in N^N$;

holds true.

Let $H \subset G^N$ consist of all sequences $U_k - (a)$, $k \in N$, and let $D \subset G^N$ consist of all sequences V_f, $f \in N^N$. Put $G_0 = \zeta\langle\delta(H \cup D)\rangle$ and for $x \in G$ put $G_x = G_0 + (x)$. Define $(S, x) \in \mathfrak{G}$ whenever $S \in G_x$. Clearly, condition $(*)$ is satisfied.

As indicated in Section 2, $\mathfrak{G} \subset G^N \times G$ satisfies axioms $(\mathscr{L}_1)$, $(\mathscr{L}_2)$, $(\mathscr{L}_3)$ and $(\mathscr{S}^*\mathscr{G})$. To verify the remaining axiom $(\mathscr{L}_0)$ of sequential convergence groups it suffices to show that (0) is the only constant sequence in G belonging to $G_0 = \zeta\langle\delta(H \cup D)\rangle$.

Suppose that $S \in G^N$ is a constant sequence belonging to $\zeta\langle\delta(H \cup D)\rangle$. Since $S \circ s = S$ for each $s \in \mathscr{S}$, we can assume that $S \in \langle\delta(H \cup D)\rangle$, i.e. $S = \sum_{k=1}^{g} w_k T_k$, where g is a nonnegative integer, $w_k \in Z$ and $T_k \in \delta(H \cup D)$. Further, there is a mapping $s \in \mathscr{S}$ such that each two sequences $T_k \circ s$ and $T_l \circ s$ are either identical or we have $(T_k \circ s)(n) \neq (T_l \circ s)(n)$ for all $n \in N$. Hence $S = \sum_{k=1}^{h} z_k S_k - (za)$, where $h \leqq g$, $z_k \in Z$, $z = \sum_{k=1}^{h'} z_k$, $h' \leqq h$, S_k is either a subsequence of U_i, $i \in N$, or a subsequence of V_f, $f \in N^N$, and $S_k(n) \neq S_l(n)$ for all $n \in N$ whenever $k \neq l$. Thus $S + (za) = \sum_{k=1}^{h} z_k S_k$ is a constant sequence in G. It follows from the definition of sequences U_i and V_f that there are natural numbers n_1 and n_2 such that $\left(\bigcup_{k=1}^{h} \operatorname{gen}(S_k(n_1))\right) \cap \left(\bigcup_{k=1}^{h} \operatorname{gen}(S_k(n_2))\right) = \emptyset$. Since G is a free Z-module and $\sum_{k=1}^{h} z_k\, S_k(n_1) = \sum_{k=1}^{h} z_k\, S_k(n_2)$, we get $z_k = 0$ for all $k = 1, \ldots, h$. Thus $S = (0)$.

Since the subspace $A \cup \{0\}$ of G is homeomorphic to the nonseparated space X (mentioned above), the proof is finished.

References

[1] *M. Dolcher:* Topologie e strutture di convergenza. Ann. della Scuola Norm. Sup. di Pisa **III**, Vol. *XIV*, Fasc. I (1960), 63—92.

[2] *J. Novák:* On some problems concerning convergence spaces and groups. General Topology and its Relations to Modern Analysis and Algebra (Proc. Kanpur Topological Conf., 1968). Academia, Praha 1970, 219—229.

[3] *F. Zanolin:* Solution of a problem of J. Novák about convergence groups. Bollettino Un. Mat. Ital. (5) *14*-A (1977), 375—381.

Author's address: Università degli Studi di Trieste, Istituto di Matematica, piazzale Europa 1, 34100 Trieste, Italy.