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# ON A GENERALIZATION OF THE MATROID

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The notation of the semimatroid is known from literature (see [2]). A semimatroid is a pair $H = \langle X, \mathscr{B} \rangle$, where $X$ is a non empty finite set and $\mathscr{B}$ is a non empty antihereditary family of subsets of $X$, which means that $\mathscr{B}$ satisfies the condition

(i) $$B \in \mathscr{B} \wedge A \subsetneqq B \Rightarrow A \notin \mathscr{B}\,.$$

The sets from $\mathscr{B}$ will be called *bases of the semimatroid H*. A semimatroid $H = \langle X, \mathscr{B} \rangle$ satisfying the condition

(e) $$B_1, B_2 \in \mathscr{B} \Rightarrow \bigwedge_{x \in B_1 \setminus B_2} \bigvee_{y \in B_2 \setminus B_1} [(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}]$$

is called a *matroid* (see [3]).

In this paper we consider a generalization of the condition (e), namely: a semimatroid $H = \langle X, \mathscr{B} \rangle$ will be called an e*-*semimatroid* if $H$ satisfies the condition

(e*) $$B_1, B_2 \in \mathscr{B} \Rightarrow \bigwedge_{x_1 \in B_1 \setminus B_2} \bigvee_{y_1 \in B_1} \bigvee_{x_2, y_2 \in B_2} [(B_1 \setminus \{x_1, y_1\}) \cup \cup \{x_2, y_2\} \in \mathscr{B} \wedge (x_1 = y_1 \Rightarrow x_2 = y_2)]\,.$$

It is easy to verify that any matroid is an e*-semimatroid. On the other hand, there are e*-semimatroids that are not matroids, which is shown by the following example.

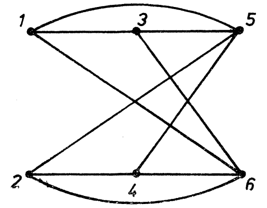


Figure 1.

Example 1. Let us take the simple graph $G$ (see [1]) in Fig. 1.
Let us consider a semimatroid $H = \langle X, \mathscr{B} \rangle$, where $X = \{1, 2, 3, 4, 5, 6\}$ and $\mathscr{B}$ is

the set of all cliques of $G$, i.e. the sets of vertices of maximal complete subgraphs of $G$. Therefore $\mathscr{B} = \{\{1, 3, 5\}, \{1, 3, 6\}, \{2, 4, 5\}, \{2, 4, 6\}\}$. Observe that $H$ satisfies (e*) but not (e). We see that some semimatroids generated by graphs can be e*-semimatroids which was the reason for the authors to consider e*-semimatroids.

In this paper we prove in Section 1 that any two bases of an e*-semimatroid have the same number of elements (Theorem 1).

In Section 2 (Theorem 2) we show the following result: *Let $G = (U, X)$ be a simple connected graph. Let $T$ be the set of edges of a spanning tree of $G$, let $T^*$ be obtained from $T$ by removing one pendant edge. Denote by $\mathscr{T}^*$ the family of all sets of the form $T^*$. Then the pair $H = \langle X, \mathscr{T}^* \rangle$ is an e*-semimatroid but not neccessarily a matroid.*

In Section 3 we give a representation of graphs in which cliques form a matroid and produce examples of graphs in which cliques form an e*-semimatroid.

## 1

Let $H = \langle X, \mathscr{B} \rangle$ be an e*-semimatroid. We shall consider the following condition:

$$(*) \qquad B_1, B_2 \in \mathscr{B}, \quad B_3 = (B_1 \setminus \{x_1, y_1\}) \cup \{x_2, y_2\} \in \mathscr{B},$$

where $\quad x_1 \in B_1 \setminus B_2, \quad y_1 \in B_1, \quad x_2, y_2 \in B_2 \quad$ and $\quad (x_1 = y_1 \Rightarrow x_2 = y_2)$.

**Lemma 1.** *If $(*)$ holds then $\{x_2, y_2\} \not\subset B_1 \cap B_2$.*

Proof. Let $x_2, y_2 \in B_1 \cap B_2$, then $B_1 \setminus \{x_1, y_1\} = B_3 \subsetneqq B_1$. Hence and by (i) $B_3 \notin \mathscr{B}$ – a contradiction.

**Lemma 2.** *Let $(*)$ hold, let $B_3 \setminus B_1 = \{y_2\}$ and $x_1 \neq y_1$, then $x_2 = y_1$.*

Proof. Suppose $x_2 \neq y_1$. Then $y_1 \notin B_3$. We apply (e*) to the bases $B_3$ and $B_1$ and to the element $y_2$. So there exist $z_2 \in B_3$ and $u_1, v_1 \in B_1$ such that $(B_3 \setminus \{y_2, z_2\}) \cup \cup \{u_1, v_1\} = B_4 \in \mathscr{B}$ and $(y_2 = z_2 \Rightarrow u_1 = v_1)$. We shall show that $B_4 \subsetneqq B_1$ thus obtaining a contradiction $B_4 \in \mathscr{B}$. Let $z_2 \in B_1$. Since $x_1, y_1 \notin B_3$ and $B_3 \setminus B_1 = \{y_2\}$ so $B_4 = (B_1 \setminus \{x_1, y_1, z_2\}) \cup \{u_1, v_1\}$. Observe that $z_2 \neq x_1$ and $z_2 \neq y_1$, since $x_1, y_1 \notin B_3$ and $z_2 \in B_3$. Moreover, $x_1 \neq y_1$, hence $B_4 \subsetneqq B_1$. Let now $z_2 \notin B_1$. Then $z_2 \in B_3 \setminus B_1$, so $z_2 = y_2$. Thus we have $u_1 = v_1$, hence $B_4 = (B_1 \setminus \{x_1, y_1\}) \cup \cup \{u_1\} \subsetneqq B_1$.

**Lemma 3.** *If $(*)$ holds then we have exactly one of the following three possibilities:*

(1) $x_1 \in B_1 \setminus B_2$, $y_1, x_2 \in B_1 \cap B_2$, $y_2 \in B_2 \setminus B_1$ *and* $x_2 = y_1$;
(2) $x_1 \in B_1 \setminus B_2$, $y_1 \in B_1 \cap B_2$, $x_2, y_2 \in B_2 \setminus B_1$ *and* $x_2 \neq y_2$;
(3) $x_1, y_1 \in B_1 \setminus B_2$, $x_2, y_2 \in B_2 \setminus B_1$ *and* $(x_1 = y_1$ *and* $x_2 = y_2$ *or* $x_1 \neq y_1$ *and* $x_2 \neq y_2)$.

Proof. Consider all the possibilities for the condition (e*):
(4) $y_1 \in B_1 \cap B_2$, $x_2, y_2 \in B_1 \cap B_2$;
(5) $y_1 \in B_1 \cap B_2$, $x_2 \in B_1 \cap B_2$, $y_2 \notin B_1 \cap B_2$;
(6) $y_1 \in B_1 \cap B_2$, $x_2, y_2 \notin B_1 \cap B_2$;
(7) $y_1 \notin B_1 \cap B_2$, $x_2, y_2 \in B_1 \cap B_2$;
(8) $y_1 \notin B_1 \cap B_2$, $x_2 \in B_1 \cap B_2$, $y_2 \notin B_1 \cap B_2$;
(9) $y_1 \notin B_1 \cap B_2$, $x_2, y_2 \notin B_1 \cap B_2$.
By Lemma 1 the cases (4), (7) cannot hold. The case (5) gives (1) by Lemma 2. The case (8) cannot hold. In fact, if $x_2 \in B_1 \cap B_2$ and $y_2 \notin B_1 \cap B_2$ then $x_2 \neq y_2$. Hence by (*) we have $B_3 \setminus B_1 = \{y_2\}$ and $x_1 \neq y_1$. In view of Lemma 2 we get $x_2 = y_1$ which is impossible. Consider the case (6). Let $y_1 \in B_1 \cap B_2$, $x_2, y_2 \notin B_1 \cap B_2$ and (*) hold. Then $x_2 \neq y_2$. In fact, if $x_2 = y_2$ then $B_3 \setminus B_1 = \{y_2\}$. Since $x_1 \neq y_1$ so by Lemma 2 we have $x_2 = y_1$ which is impossible. So we have the possibility (2).

Consider the case (9). Let $y_1, x_2, y_2 \notin B_1 \cap B_2$ and (*) hold. If $x_1 = y_1$ then $x_2 = y_2$ by (*). Suppose that $x_1 \neq y_1$ and $x_2 = y_2$. Then $B_3 \setminus B_1 = \{y_1\}$ and by Lemma 2 $x_2 = y_1$ which cannot hold. So we have $x_1 \neq y_1 \Rightarrow x_2 \neq y_2$. In the case (9) we have the possibility (3).

**Corollary 1.** *If (*) holds then* $|B_3| = |B_1|$.

Proof. It follows from Lemma 3 that the number of elements rejected from $B_1$ is equal to the number of elements added to $B_1$.

**Theorem 1.** *Any two bases of an e*-semimatroid have the same number of elements.*

Proof. Let $H = \langle X, \mathscr{B} \rangle$ be an e*-semimatroid and $B_1, B_2 \in \mathscr{B}$. If $B_1 \setminus B_2 \neq \emptyset$ and $x_1 \in B_1 \setminus B_2$ then we can form the basis $B_3$ as in (*) and by Corollary 1 we have $|B_3| = |B_1|$. By Lemma 3 we obtain that the basis $B_3$ arises by deleting at least one element from the set $B_1 \setminus B_2$ and adding at least one element from the set $B_2 \setminus B_1$. If $B_3 \setminus B_2 \neq \emptyset$ then we can form the basis $B_4$ for the bases $B_3$ and $B_2$ analogously as we formed the basis $B_3$ for the bases $B_1$ and $B_2$ in (*).

Then by Corollary 1 we get $|B_4| = |B_3| = |B_1|$. Observe that $B_3 \setminus B_2 \subset B_1 \setminus B_2$ and $B_4$ arises by deleting at least one new element from the set $B_1 \setminus B_2$ and adding at least one new element from the set $B_2 \setminus B_1$. After a finite number of steps we get a basis $B_k$ such that $B_k \subset B_2$ and $|B_k| = |B_{k-1}| = |B_{k-2}| = \ldots = |B_3| = |B_1|$. By (i) we have $B_k = B_2$ which completes the proof.

## 2

Let $G = (U, X)$ be a simple connected graph. It is known that a pair $\langle X, \mathscr{T} \rangle$, where $\mathscr{T}$ is the set of all spanning trees of $G$, is matroid (see [1]). L. Szamkołowicz asked which subsets of $X$ form an e*-semimatroid. We answer this question in the following theorem.

**Theorem 2.** *Let $G = (U, X)$ be a simple connected graph. Let $T$ be the set of edges of a spanning tree of $G$, and let $T^*$ be obtained from $T$ by removing one pendant edge. Let $\mathscr{T}^*$ denote the family of all sets of the form $T^*$. Then the pair $H = \langle X, \mathscr{T}^* \rangle$ is an e*semimatroid but not necessarily a matroid.*

Proof. If $|U| \leqq 3$ then any spanning tree has at most two edges and any $T^*$ has at most one edge and (e*) is satisfied. Suppose $|U| \geqq 4$. Let $T^*$ be obtained from $T$ by removing a pendant edge. Denote the removed edge by $p(T)$. Denote by $i(T^*)$ the vertex of $p(T)$ which becomes an isolated vertex after removing $p(T)$ from the tree $T$. Observe that $\mathscr{T}^*$ satisfies the condition (i) of the semimatroid. We shall show that (e*) holds. Let $T_1^*, T_2^* \in \mathscr{T}^*$ and $x_1 \in T_1^*$. The graph $(U, T_1^* \setminus \{x_1\})$ has three components $i(T_1^*), K_1, K_2$.

If there exists $x_2 \in T_2^*$ such that $x_2 = \{v_1, v_2\}$ and $v_1 \in K_1$, $v_2 \in K_2$ then $(T_1^* \setminus \{x_1\}) \cup \{x_2\} = T_3^* \in \mathscr{T}^*$ since $T_3^* \cup \{p(T_1)\} \in \mathscr{T}$. If such an edge $x_2$ does not exist then necessarily $i(T_1^*) \neq i(T_2^*)$ since otherwise we have three components in the graph $(U, T_2^*)$. If $|K_1| = 1$ and $K_1 = \{i(T_2^*)\}$ then there exists in $T_2^*$ an edge $x_2 = \{u, i(T_1^*)\}$ with $u \in K_2$. Hence $(T_1^* \setminus \{x_1\}) \cup \{x_2\} \in \mathscr{T}^*$. We have the analogous situation if $K_2 = \{i(T_2^*)\}$. In the remaining case there exist in $T_2^*$ edges $\{u, i(T_1^*)\}$ and $\{i(T_1^*), v\}$ with $u \in K_1$, $v \in K_2$. Putting $x_2 = \{u, i(T_1^*)\}$, $y_2 = \{i(T_1^*), v\}$ and taking for $y_1$ an arbitrary pendant edge of $T_1^*$ different from $x_1$ we obtain $T_3^* = (T_1^* \setminus \{x_1, y_1\}) \cup \{x_2, y_2\}$. Such an edge $y_1$ exists since one of the components $K_1, K_2$ has more than one vertex. Obviously $T_3^* \in \mathscr{T}^*$ as $(T_1 \setminus \{x_1\}) \cup \{x_1, y_2\}$ is a spanning tree.

Consider now the graph in Fig. 2.

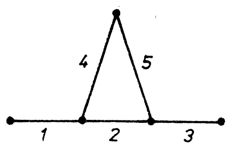


Figure 2.

Denote $T_1^* = \{1, 2, 3\}$, $T_2^* = \{3, 4, 5\}$. For $x_1 = 2$ there does not exist $x_2 \in T_2^* \setminus T_1^*$ such that $(T_1^* \setminus \{x_1\}) \cup \{x_2\} \in \mathscr{T}^*$. This proves the second part of Theorem 2.

## 3

Now we shall consider the problem when the cliques of a graph form a matriod or an e*-semimatroid.

It is known that

($\alpha$) if $\langle X, \mathscr{B} \rangle$ is a matroid then the following condition is satisfied:

$$\bigwedge_{B_1, B_2 \in \mathscr{B}} \bigwedge_{y \in B_2 \setminus B_1} \bigvee_{x \in B_1 \setminus B_2} [(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}] .$$

Let $G = (U, X)$ be a simple graph. We denote the set of the cliques of $G$ by $\mathscr{K}_G$.

**Theorem 3.** *The pair $H = \langle U, \mathscr{K}_G \rangle$ is a matroid iff there exists a partition $\{U_i\}_{i \in I}$ of the set U such that*

$$(**) \qquad \bigwedge_{u \in U_s} \bigwedge_{v \in U_t} [\{u, v\} \in X \Leftrightarrow s \neq t] \quad \text{where} \quad s, t \in I ;$$

*in other words G is a complete I-partite graph.*

Proof. Sufficiency is obvious since the only cliques of $G$ are the sets of the form $B^*$ where $\bigwedge_{i \in I} |B^* \cap U_i| = 1$.

Proof of necessity. Let $H = \langle U, \mathscr{K}_G \rangle$ be a matroid and $K_0$ a fixed basis of $H$, i.e. a fixed clique of $G$. For every $a \in K_0$ we define the set $U_a = \{u \in U; (K_0 \setminus \{a\}) \cup \cup \{u\} \in \mathscr{K}_G\}$. We shall show that the family $\{U_0\}_{a \in U}$ is a partition of $U$. Obviously for any $a \in K_0$ we have $U_a \neq \emptyset$ since $a \in U_a$. If $a \neq b$, $a, b \in K_0$ then $U_a \cap U_b = \emptyset$.

Suppose on the contrary that a vertex $u$ in the graph $G$ is connected with all vertices of the clique $K_0$. This, however, contradicts the maximality of $K_0$.

$$\bigwedge_{u \in U} \bigvee_{a \in K_0} u \in U_a .$$

In fact, if $u \in K_0$, then $u \in U_u$. Let $u \in U \setminus K_0$. Obviously $\{u\}$ can be extended to the maximal complete subgraph $M$ of the graph $G$. So $M$ is a clique and $M \in \mathscr{K}_G$. Applying Lemma 4 with $y = u$ and the cliques $K_0$ and $M$ we find $a \in K_0$ such that $(K_0 \setminus \{a\}) \cup \{u\}$ is a clique, hence $u \in U_a$.

Now we shall show the validity of the condition $(**)$. We shall show

$$(10) \qquad \text{if} \quad u \in U_s, \quad v \in U_t, \quad \{u, v\} \in X \quad \text{then} \quad s \neq t .$$

Suppose $u \in U_s$, $v \in U_t$, $\{u, v\} \in X$ and $s = t$. Since $u \in U_s$, so $(K_0 \setminus \{s\}) \cup \{u\} \in \mathscr{K}_G$. Analogously $(K_0 \setminus \{s\}) \cup \{v\} \in \mathscr{K}_G$. Hence $(K_0 \setminus \{s\}) \cup \{u, v\}$ is a complete subgraph and it can be extended to some clique $K$. Then $(K_0 \setminus \{s\}) \cup \{u\} \subsetneqq K$ which contradicts the maximality of the clique $(K_0 \setminus \{s\}) \cup \{u\}$. Let now $u \in U_s$, $v \in U_t$ and $s \neq t$. Then $K_1 = (K_0 \setminus \{s\}) \cup \{u\} \in \mathscr{K}_G$, $K_2 = (K_0 \setminus \{t\}) \cup \{v\} \in \mathscr{K}_G$. We exchange the cliques $K_1, K_2$ and the element $t \in K_1 \setminus K_2$. Observe that $K_2 \setminus K_1 = \{v, s\}$. If $K_1 \setminus \{t\} \cup \{s\} \in \mathscr{K}_G$ then $\{s, u\} \in X$ contrary to (10).

Now we shall show a class of graphs in which cliques form e*-semimatroids but not necessarily matroids.

Let $n_1, \ldots, n_r$ be a sequence of positive integers such that $n_i \geqq n_j$ for $i < j$. Consider a graph $G_{(n_1, \ldots, n_r)} = (U_1 \cup U_2 \cup \ldots \cup U_r, X)$ where $U_i = \{a_{i1}, \ldots, a_{in_i}\}$, $U_i \cap U_j \in \emptyset$ for $i \neq j$, $i, j \in \{1, \ldots, r\}$. The set $X$ contains all possible edges except those of the following three forms:

$$(11) \qquad \{a_{ik_1}, a_{ik_2}\} \text{ for } k_1, k_2 \in \{1, \ldots, n_i\} ,$$

$$(12) \qquad \{a_{(2s-1),k}, a_{2s,k}\} \text{ where } k' \neq k \text{ and } 1 \leqq k, k' \leqq n_{2s},$$

$$(13) \qquad \{a_{(2s-1),t}, a_{2s,t}\} \text{ where } n_{2s} < t \leqq n_{2s-1},\ 1 \leqq t' \leqq n_{2s},\ 1 \leqq s \leqq [\tfrac{1}{2}r].$$

Observe that the intersection of any clique of the graph $G_{(n_1,\ldots,n_r)}$ with any of the sets $U_i$ ($i = 1, \ldots, r$) contains exactly one element. Checking that the condition (e*) holds for the clique is easy.

The following problem is open:

*Describe all simple graphs in which cliques form e*-semimatroids.*

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