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SUBSPACES OF  $L_\infty(G)$  WITH UNIQUE TOPOLOGICAL  
LEFT INVARIANT MEAN

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## 1. INTRODUCTION

In what follows we denote by  $G$  always a locally compact Hausdorff group with left invariant Haar measure. Let  $A$  be an  $L_1(G)$ -submodule of  $L_\infty(G)$  which is left invariant and containing the constant functions. A mean on  $A$  is a linear functional  $m$  on  $A$  such that  $m(\bar{g}) = \overline{m(g)}$  for all  $g \in A$  (the bar denoting complex conjugation),  $m(1) = 1$ , and  $m(g) \geq 0$  if  $g \geq 0$  locally almost everywhere. A mean  $m$  on  $A$  is called *left invariant* (LIM) if  $m_a(g) = m(g)$  for all  $a$  in  $G$  and all  $g$  in  $A$ . A *topologically left invariant* mean (TLIM) on  $A$  is a mean  $m$  such that  $m(\varphi * g) = m(g)$  for all  $g \in A$  and all  $\varphi \in P(G) = \{\varphi \in L_1(G) : \varphi \geq 0, \|\varphi\|_1 = 1\}$ .

It is well known (see e.g. [1], [6] and [8]) that on each of the spaces  $AP(G)$  and  $W(G)$ , being respectively the sets of almost periodic and weakly almost periodic functions in  $L_\infty(G)$  there exists a unique LIM; and it is also the unique TLIM. In section 2 we construct two new subspaces of  $L_\infty(G)$ , one of them containing properly  $AP(G)$  and the other  $W(G)$ , such that on each of these new spaces there exists a unique TLIM. For Abelian  $G$  with dual  $\hat{G}$  the first space coincides precisely with the space of those functions which are almost periodic at every point of  $\hat{G}$ , as introduced by Loomis in [7].

All of these results are shown by use of the so-called  $\tau_c$ - and  $\tau_w$ -topologies, which have been introduced in [2] and [3]. For convenience we repeat here their definitions. The space  $L_\infty(G)$  may be embedded into  $B(L_1(G), L_\infty(G))$  by the operator  $\Phi$  such that  $\Phi(g)(f) = f * g$  ( $f \in L_1(G)$ ,  $g \in L_\infty(G)$ ,  $*$  the convolution product). Since  $B(L_1(G), L_\infty(G))$  carries naturally the strong and the weak operator topology,  $\Phi$  allows us to consider their induced topologies on  $L_\infty(G)$ , which we denote by  $\tau_c$  and  $\tau_w$  respectively. These topologies may also be introduced in another manner; indeed, each  $f \in L_1(G)$  induces by convolution an operator  $C_f$  on  $L_\infty(G)$  which is continuous when  $L_\infty(G)$  carries its norm topology  $\|\cdot\|_\infty$ ; the weak topology on  $L_\infty(G)$  under the convolution operators  $C_f : L_\infty(G) \rightarrow (L_\infty(G), \|\cdot\|_\infty)$  then coincides with  $\tau_c$ , while  $\tau_w$  is the weak topology on  $L_\infty(G)$  under the same set of operators  $C_f : L_\infty(G) \rightarrow (L_\infty(G), w)$ , where  $w$  denotes the weak topology on  $L_\infty(G)$ . So we

immediately obtain  $w^* \leq \tau_w \leq \tau_c \leq \|\cdot\|$ , and  $w^* \leq \tau_w \leq w \leq \|\cdot\|_\infty$ . Moreover,  $\tau_c \equiv \|\cdot\|_\infty$  iff  $G$  is discrete.

All other nonexplained notations and definitions are taken from [8].

## 2. SUBSPACES OF $L_\infty(G)$ WITH UNIQUE TLIM

We start with the following lemma.

**Lemma 2.1.** *Let  $A$  be an  $L_1(G)$ -submodule of  $L_\infty(G)$ . A LIM  $m$  on  $A$  is a TLIM iff  $m$  is continuous for the induced  $\tau_c$ -topology.*

*Proof.* Let  $m$  be a TLIM on  $A$ . If  $g$  is a fixed function in  $A$  and  $(g_\lambda)_{\lambda \in A}$  is a net in  $A$  that  $\tau_c$ -converges to  $g$ , then the net  $(\varphi * g_\lambda)_{\lambda \in A}$  is  $\|\cdot\|_\infty$ -convergent to  $\varphi * g$ , for each  $\varphi \in P(G)$ .

Since  $m$  is always continuous for the  $\|\cdot\|_\infty$ -topology, the result follows from the fact that  $m(\varphi * h) = m(h)$  for all  $\varphi \in P(G)$  and all  $h \in A$ .

Conversely, let  $m$  be a LIM on  $A$  which is  $\tau_c$ -continuous. Using the left invariance of  $m$  we obtain that  $m(a * f * g) = m(f * g)$  for all  $a \in G, f \in L_1(G), g \in A$ . In particular, the functional  $f \rightarrow m(f * g)$  on  $L_1(G)$  is linear, bounded, and left invariant, and so there exists a constant (depending on  $g$ ), say  $c(g)$ , such that  $m(f * g) = c(g) \int_G f(t) dt$  for all  $f \in L_1(G)$ ; this leads to  $m(\varphi * g) = c(g)$  for  $\varphi \in P(G)$ . Let then  $(e_\lambda)_{\lambda \in A}$  be an approximate identity in  $L_1(G)$  such that each  $e_\lambda$  belongs to  $P(G)$ . For  $g$  in  $A$ , the net  $(e_\lambda * g)_{\lambda \in A}$  is  $\tau_c$ -convergent to  $g$ . So, due to the  $\tau_c$ -continuity of  $m$  we obtain  $c(g) = m(e_\lambda * g) \rightarrow m(g)$ , while due to the  $\|\cdot\|_\infty$ -continuity of  $m$  we also have  $c(g) = m((\varphi * e_\lambda) * g) \rightarrow m(\varphi * g)$ , for all  $\varphi \in P(G)$ . Hence  $m$  is a TLIM on  $A$ . ■

We now construct Banach subspaces of  $L_\infty(G)$  on which there exists a unique TLIM.

To this end, call a function  $g$  in  $L_\infty(G)$  *right almost periodic with respect to  $\tau_c$*  ( $r - \tau_c - \text{a.p.}$ ) iff the set  $\{g_a : a \in G\}$  of right translates of  $g$  is relatively compact with respect to  $\tau_c$ . We denote the set of these functions by  $R - \tau_c - \text{AP}$ . Analogously, using the  $\tau_w$ -topology we may define the set  $R - \tau_w - \text{AP}$ . Since the spaces  $(L_\infty(G), \tau_c)$  and  $(L_\infty(G), \tau_w)$  are Hausdorff topological vector spaces, it may be verified that both sets are right invariant linear subspaces of  $L_\infty(G)$ .

**Lemma 2.2.**

$$\begin{aligned} g \in R - \tau_c - \text{AP} &\Leftrightarrow f * g \in AP(G), \quad \forall f \in L_1(G), \\ g \in R - \tau_w - \text{AP} &\Leftrightarrow f * g \in W(G), \quad \forall f \in L_1(G). \end{aligned}$$

*Proof.* We only give the proof of the first equivalence. Since for any  $f \in L_1(G)$ , each operator  $C_f : (L_\infty(G), \tau_c) \rightarrow (L_\infty(G), \|\cdot\|_\infty)$  with  $C_f(g) = f * g$  is continuous, one implication is quickly verified using the fact that  $(f * g)_a = f * g_a$ .

To prove the inverse implication, let  $\Phi : L_\infty(G) \rightarrow B(L_1(G), L_\infty(G))$  be the operator defined in the introduction, and put  $A = \{(\Phi(g))_a : a \in G\}$ , where we define

$(\Phi(g))_a(f) = (f * g)_a = \Phi(g_a)(f)$ . Then  $A \subset B(L_1(G), L_\infty(G))$ , and an adaptation of exercise VI.9.2 in [4] shows that  $A$  is relatively compact in the strong operator topology. The result then follows from the definition of  $\tau_c$ .

The proof of the second equivalence is analogous. ■

From lemma 2.2 we derive that both sets  $R - \tau_c - AP$  and  $R - \tau_w - AP$  are  $\tau_c$ -closed. Indeed, if  $(g_\lambda)_{\lambda \in A}$  is a net in one of these sets such that  $(g_\lambda)_{\lambda \in A}$   $\tau_c$ -converges to  $g$ , then the net  $(f * g_\lambda)_{\lambda \in A}$ , which is in either  $AP(G)$  or  $W(G)$ , is  $\|\cdot\|_\infty$ -convergent to  $f * g$ , for each  $f$  in  $L_1(G)$ . Since both sets  $AP(G)$  and  $W(G)$  are  $\|\cdot\|_\infty$ -closed, the limit function  $g$  also belongs to either  $R - \tau_c - AP$  or  $R - \tau_w - AP$ .

Of course  $R - \tau_c - AP$  and  $R - \tau_w - AP$  are also  $\|\cdot\|_\infty$ -closed (hence they are Banach subspaces) since  $\tau_c \leq \|\cdot\|_\infty$ ; being convex sets, they are also  $\tau_w$ -closed.

In order to obtain our next result, we state [2, coroll. 3 and 4] in the form of the following lemma;  $\text{cl}_\tau B$  denotes the closure of a set  $A$  in the topology  $\tau$ .

**Lemma 2.3.** *Let  $S$  be a  $\tau_c$ -closed  $L_1(G)$  submodule of  $L_\infty(G)$ . Then  $S = \text{cl}_{\tau_c}(L_1(G) * S)$ , and  $S$  is left translation invariant.*

Since  $AP(G) \subset R - \tau_c - AP$ , and due to the fact that  $L_1(G) * AP(G) = AP(G)$ , we have from lemma 2.2  $AP(G) = L_1(G) * AP(G) \subset L_1(G) * R - \tau_c - AP \subset AP(G)$ . Hence  $L_1(G) * R - \tau_c - AP = AP(G)$ , and from lemma 2.2 we derive that  $R - \tau_c - AP = \text{cl}_{\tau_c}(AP(G))$ . Analogously,  $L_1(G) * R - \tau_w - AP = W(G)$ , and  $R - \tau_w - AP = \text{cl}_{\tau_c}(W(G))$ . Moreover, both sets  $R - \tau_c - AP$  and  $R - \tau_w - AP$  are left invariant.

**Theorem 2.4.** *There exists a unique TLIM on  $R - \tau_c - AP$ .*

*Proof.* There exists a unique LIM  $m$  on  $AP(G)$ , and it is also a TLIM; hence  $m$  is also continuous for the induced  $\tau_c$ -topology. Since  $R - \tau_c - AP = \text{cl}_{\tau_c}(AP(G))$ , there exists an extension of  $m$  to a linear functional  $M$  on  $R - \tau_c - AP$  which is  $\tau_c$ -continuous; this extension is then necessarily unique. It remains to show that this extension  $M$  is a left invariant mean on  $R - \tau_c - AP$ . That  $M(1) = 1$ ,  $M(\bar{g}) = \overline{M(g)}$ , and  $M(ag) = M(g)$  for  $g \in R - \tau_c - AP$  is readily verified using the definition of the  $\tau_c$ -topology and the properties of  $m$ . If  $g \in R - \tau_c - AP$  and  $g \geq 0$  locally almost everywhere, choose an approximate identity  $(e_\lambda)_{\lambda \in A}$  in  $L_1(G)$  consisting of positive functions, and put  $g_\lambda = e_\lambda * g$ . Then each  $g_\lambda$  belongs to  $AP(G)$ ,  $g_\lambda \geq 0$ , and  $(g_\lambda)$   $\tau_c$ -converges to  $g$ . Hence  $M(g) \geq 0$ . Due to lemma 2.1,  $M$  is a TLIM on  $R - \tau_c - AP$ . ■

Completely analogous to theorem 2.4 we may prove

**Theorem 2.5.** *There exists a unique TLIM on  $R - \tau_w - AP$ .*

**Corollary 2.6.** *If  $G$  is compact, there exists a unique TLIM on  $L_\infty(G)$ .*

*Proof.* Since for given  $g$  in  $L_\infty(G)$  the function  $s \rightarrow g_s$  from  $G$  to  $L_\infty(G)$  is con-

tinuous for the  $\tau_c$ -topology on  $L_\infty(G)$ , any  $g \in L_\infty(G)$  is  $R - \tau_c$ -a.p. when  $G$  is compact, i.e.  $R - \tau_c - \text{AP} = L_\infty(G)$ . The result then follows from theorem 2.4. ■

**Remark 1.** Since the LIM on  $AP(G)$  or  $W(G)$  is also right invariant, the same is true for the TLIM on  $R - \tau_c - \text{AP}$  and  $R - \tau_w - \text{AP}$ .

Let  $G$  be an Abelian group with dual  $\hat{G}$ . A bounded measurable function  $g$  on  $G$  is called *almost periodic at the point*  $\gamma_0 \in \hat{G}$  iff there exists a function  $f$  in  $L_1(G)$  such that  $f * g$  is ( $\| \cdot \|_\infty$ -)almost periodic and  $\hat{f}(\gamma_0) \neq 0$  (see Loomis [7], p. 364).

**Theorem 2.7.** *For Abelian  $G$  and  $g \in L_\infty(G)$  we have  
 $g \in R - \tau_c - \text{AP}$  iff  $g$  is almost periodic at each point of  $\hat{G}$ .*

**Proof.** By lemma 2.2 it is clear that any  $g$  in  $R - \tau_c - \text{AP}$  is almost periodic at each point of  $\hat{G}$ .

To prove the converse implication we have to show that, given  $g$  in  $L_\infty(G)$  which is almost periodic at each point of  $\hat{G}$ , the function  $f * g$  belongs to  $AP(G)$  for each  $f$  in  $L_1(G)$ . We use the notation of [7]; in particular, we denote by  $\text{spg}$  the spectrum of a bounded function  $g$ . Given  $\varepsilon > 0$  and  $f$  in  $L_1(G)$ , there exists a function  $v$  in  $L_1(G)$  such that  $\hat{v}$  has compact support, and  $\|f - v * f\|_1 < \varepsilon$ ; also  $\text{sp}(v * f) \subset \subset \text{sp}v = \text{supp } \hat{v}$ . This means that there exists a net  $(h_\lambda)_{\lambda \in A}$  in  $L_1(G)$  such that  $(h_\lambda) \| \cdot \|_1$ -converges to  $f$ , while each  $h_\lambda$  has compact spectrum.

Since  $(h_\lambda * g)$  is  $\| \cdot \|_\infty$ -convergent to  $f * g$ , this function will belong to  $AP(G)$  as soon as each  $h_\lambda * g$  is almost periodic. So it suffices to prove : given  $f$  in  $L_1(G)$  with compact spectrum, then the function  $h = f * g$  is almost periodic. By [7] theorem 1, this will be the case iff  $h$  is almost periodic at each point of  $\hat{G}$ . Given  $\gamma_0 \in \hat{G}$ , there exists a function  $f_0$  in  $L_1(G)$  such that  $f_0 * g$  is almost periodic and  $\hat{f}_0(\gamma_0) \neq 0$ ; then  $f_0 * h = f * (f_0 * g)$ , and this is almost periodic since  $L_1(G) * AP(G) = AP(G)$ . ■

### 3. THE EXTENT OF $R - \tau_w - \text{AP}$

**Theorem 3.1.** *Let  $G$  be a non-compact  $\sigma$ -compact amenable group. Then the quotient space  $L_\infty(G) /_{R - \tau_w - \text{AP}}$  is nonseparable.*

**Proof.** Put  $R - \tau_w - \text{AP} \equiv A$  for short, and suppose that  $L_\infty(G) /_A$  is separable. Then there exists a countable dense subset  $\{[g_n]\}_{n=1}^\infty$  in  $L_\infty(G) /_A$ , where  $[g_n] = g_n + A$ , and  $g_n \in L_\infty(G)$ . Let  $B$  be the linear span in  $L_\infty(G)$  of the sequence  $\{g_n\}_{n=1}^\infty$ ; then  $A + B$  is dense in  $L_\infty(G)$ . Let  $m$  be a TLIM on  $L_\infty(G)$ , and put  $m(g_n) = \alpha_n$ . If  $M$  is also a TLIM on  $L_\infty(G)$  such that  $M(g_n) = \alpha_n$ , then  $M = m$ ; indeed,  $M = m$  on  $B$  by assumption, and  $M = m$  on  $A$  since  $A$  has a unique TLIM; the result then follows from the denseness of  $A + B$  in  $L_\infty(G)$ . Putting  $C = \text{cl}_w * P(G) \cap \{ \mathcal{M} \in \text{TLIM} : \mathcal{M}(g_n) = \alpha_n \}$ , we derive that  $C$  is norm separable. According to [5, theorem 5], this is sufficient to conclude that  $G$  would be compact. ■

**Corollary.** *If  $G$  is  $\sigma$ -compact and  $R - \tau_w - \text{AP} = L_\infty(G)$ , then  $G$  is compact.*

**Remark.** The result of this last corollary is also true without the assumption that  $G$  is  $\sigma$ -compact. Indeed, if  $R - \tau_w - \text{AP} = L_\infty(G)$ , then  $W(G) = L_1(G) * R - \tau_w - \text{AP} = L_1(G) * L_\infty(G) = C_{ru}(G)$ , where  $C_{ru}(G)$  denotes the set of right uniformly continuous functions on  $G$ ; hence  $W(G)$  contains the set of functions on  $G$  which are both left and right uniformly continuous. This is known to be a sufficient condition for the compactness of  $G$ .

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