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OSCILLATING AND ASYMPTOTIC PROPERTIES OF A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MAXIMA

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The present paper deals with some asymptotic and oscillating properties of functional differential equations of the form

(1) $\begin{aligned} x''(t) + \lambda x''(t-\tau) + f(t, \max_{s \in M(t)} x(s), \max_{s \in M(t)} x'(s)) &= 0, \\ \text{where } \tau > 0 \text{ is a constant delay, } M(t) &\subseteq [t_0, +\infty) \text{ when } t \in [t_0, +\infty), \ t_0 \in \mathbb{R}^1, \end{aligned}$

and $\lambda > 0$ is an arbitrary constant.

Definition 1. As a solution of equation (1) we shall consider every function $x(t) \in$ $\in \tilde{C}^{1}(I, \mathbb{R}^{1}), I = [t_{0} - \tau, +\infty), t_{0} \in \mathbb{R}^{1}$ which satisfies (1) almost everywhere when $t \ge t_0$. (We shall denote by $\tilde{C}^k(I, \mathbb{R}^1)$ the space of the functions $\varphi(t): I \mapsto \mathbb{R}^1$ possessing absolutely continuous derivatives up to order k inclusively.)

Definition 2. We shall call a continuous function $\varphi(t): I \mapsto \mathbb{R}^1$ oscillating if it contains a sequence of zeros approaching $+\infty$. Otherwise, the function will be called non-oscillating.

Definition 3. We shall call a continuous function $\varphi(t) : I \mapsto \mathbb{R}^1$ strongly oscillating if there exists a sequence of points $\{t_i\}_{i=1}$ such that $\lim_{i \to \infty} t_i = +\infty$ and $\varphi(t_i) \varphi(t_{i+1}) < 0$ $i \rightarrow +\infty$ < 0 for every *i*. Otherwise, the function $\varphi(t)$ will be called *strongly non-oscillating*.

Definition 4. [1], [2]. We shall call a continuous function $\varphi(t): I \mapsto \mathbb{R}^{\perp} k(\varphi)$ strongly oscillating $(k(\varphi) - oscillating)$ if there exists a real number $k(\varphi)$ such that the function $\varphi(t) - k(\varphi)$ is strongly oscillating (oscillating).

Definition 5. We shall call a solution x(t) of (1) correct if for every $t \in I$

$$\sup_{[t,+\infty)} |x(s)| > 0 \; .$$

Let us consider the following example:

(2)
$$x''(t) + x''(t - \pi) + \max_{s \in [t - \pi, t + \pi]} x(s) = 0,$$

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 $\tau = \pi$, $M(t) = [t - \pi, t + \pi]$. It is immediately verifiable that equation (2) has a solution $x(t) = \sin t - 1$, which is oscillating but is not strongly oscillating. On the other hand, by virtue of Theorem 1 of [2], the equation

$$x''(t) + x''(t - \pi) + x(t - \pi) = 0$$

has only strongly oscillating solutions, which shows that the maximum influences the asymptotic behaviour of functional differential equations of the neutral type.

Lemma 1. Let the following conditions hold:

1. The function $\varphi(t): I \mapsto \mathbb{R}^1$ is continuous.

2. The function $\varphi(t) + \lambda \varphi(t - \tau) \ge c \ (\varphi(t) + \lambda \varphi(t - \tau) \le -c)$ when $t \in I$, where c, τ and λ are arbitrary positive constants.

Let $s \in I$. Then for the set

$$A = \{t \mid s \leq t \leq s + 2\tau, \ \varphi(t - \tau) \geq \beta > 0\}$$
$$(A = \{t \mid s \leq t \leq s + 2\tau, \ \varphi(t - \tau) \leq -\beta < 0\})$$

the inequality mess $A \ge \tau$ holds, where β is a constant depending solely on c, τ and λ .

The proof of Lemma 1 is given in [2].

We introduce the following notations:

(3)

$$(Lx)(t) := x(t) + \lambda x(t - \tau),$$

$$f^{*}(t, u_{0}) = \inf_{\substack{v \in \mathbb{R}^{1} \\ |u| > u_{0}}} |f(t, u, v)| \quad \text{when} \quad u_{0} > 0,$$

$$M^{0}(t) = \min_{\substack{s \in M(t) \\ s \in M(t)}} s.$$

We shall say that the conditions (A) hold if the following conditions are satisfied:

A1. The function $f(t, u, v): I \times \mathbb{R}^2 \mapsto \mathbb{R}^1$ is continuous and $f(t, 0, 0) \equiv 0$ when $t \in I$.

A2. If $u \neq 0$, then u f(t, u, v) > 0 when $t \in I$.

A3. The set M(t) is closed when $t \in I$ and $\lim_{t \to +\infty} M^0(t) = +\infty$.

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Lemma 2. Let the following be fulfilled:

1. Conditions (A) hold.

2. For every constant c > 0, the identity

(4)
$$\int_{t_0}^{+\infty} f^*(t, c) dt = +\infty$$

holds.

Then every non-oscillating solution of equation (1) satisfies $\lim |x(t)| = 0$.

Proof. Let us assume that there exists a non-oscillating solution of the equation

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(1) with the property $\liminf_{t \to +\infty} |x(t)| \ge c > 0$ and, to be more precise, let us assume that x(t) < 0 when $t \ge t^*$, $t^* \in I$. By integrating (1) from t^* to $t > t^*$, we obtain

(5)
$$[(Lx)(t)]' - [(Lx)(t^*)]' = -\int_{t^*}^t f(z, \max_{s \in M(z)} x(s), \max_{s \in M(z)} x'(s) dz \ge 0.$$

If we assume that $[(Lx)(t)]' \ge c_1 > 0$ when $t \ge t^*$, then the following inequality holds when $t \ge t^*$:

$$(Lx)(t) - (Lx)(t^*) \ge c_1(t - t^*),$$

whence it follows that (Lx)(t) > 0 for sufficiently large values of t, which contradicts the assumption that x(t) < 0.

Therefore $[(Lx)(t)]' \leq 0$ when $t \geq t^*$ and, taking into account that the function [(Lx)(t)]' is monotone increasing, we conclude that the integral present on the right-hand side of the equality (5) is convergent.

On the other hand, it follows from the fact that $\limsup_{t \to +\infty} x(t) \leq -c < 0$ that there exists a point $\overline{t} \geq t^*$ such that $x(t) \leq -c/2$ when $t \geq \overline{t}$. Hence, it follows from condition A3 that there exists a point $t_1 \geq \overline{t}$ such that $\max_{s \in M(t)} x(s) \leq -c/2$ when $t \geq t_1$.

Consequently, from (3) we obtain

$$\left| \int_{t_1}^{+\infty} f(t, \max_{s \in M(t)} x(s), \max_{s \in M(t)} x'(s)) \, \mathrm{d}t \right| \ge \int_{t_1}^{+\infty} f^*(t, c/2) \, \mathrm{d}t$$

and hence

$$\int_{t_1}^{+\infty} f^*(t, c/2) \,\mathrm{d}t < +\infty ,$$

which contradicts equality 4. Thus Lemma 2 has been proved.

Theorem 1. Let the following be fulfilled:

- 1. Conditions (A) hold.
- 2. Condition 2 of Lemma 2 holds.
- 3. When $t \in I$ the inequality

mess
$$(M(t) \cap [t, t + 2\tau]) \ge \tau$$

holds.

Then equation (1) does not admit correct non-negative strongly non-oscillating solutions.

Proof. Let us assume that there exists a correct strongly non-oscillating nonnegative solution x(t) of equation (1), $x(t) \ge 0$ when $t \ge t^*$, $t^* \in I$. It follows from equation (1) that [(Lx)(t)]' is a monotone decreasing and non-negative function and hence from (5) we obtain

$$\int_{t^*}^{+\infty} f(t, \max_{s \in M(t)} x(s), \max_{s \in M(t)} x'(s)) dt < +\infty.$$

Since $[(Lx)(t)]' \ge 0$ when $t \ge t^*$, and x(t) is a correct solution, there exists a point $\overline{t} \ge t^*$ such that, when $t \ge \overline{t}$, the inequality $(Lx)(t) \ge c > 0$ holds.

Then, by virtue of Lemma 1, there exists a closed measurable set $E \subseteq [\bar{t}, +\infty)$ such that when $t \ge \bar{t}$ the inequality mess $(E \cap [t, t + 2\tau]) \ge \tau$ holds, and $x(t) \ge \beta > 0$ when $t \in E$. It follows from condition A3 that there exists a point $t_1 \ge \bar{t}$ such that $M(t) \subseteq [\bar{t}, +\infty)$ when $t \ge t_1$.

Let $t \ge t_1$ be an arbitrary point. Then, since the interval $[t, t + 2\tau]$ is a connected set and $E \cap [t, t + 2\tau]$ and $M(t) \cap [t, t + 2\tau]$ are closed sets, it follows from condition 3 of the theorem that

$$E \cap M(t) \cap [t, t + 2\tau] \neq \emptyset$$
.

Therefore, since $x(t) \ge \beta$ for $t \in E$, we can assert that for $t \ge t_1$ the inequality $\max_{s \in M(t)} x(s) \ge \beta$ holds. Using the inequality

$$\int_{t_1}^{+\infty} f(t, \max_{s \in M(t)} x(s), \max_{s \in M(t)} x'(s)) dt \ge \int_{t_1}^{+\infty} f^*(t, \beta) dt$$

we obtain

$$\int_{t_1}^{+\infty} f^*(t,\beta) \,\mathrm{d}t < +\infty \;,$$

which contradicts equality (4).

Thus Theorem 1 has been proved.

Theorem 1 proves that the maximum, in a sense, is a generator of oscillations for equations of the neutral type.

Theorem 2. Let the conditions of Theorem 1 be fulfilled.

Then each correct solution x(t) of equation (1) is k(x) strongly oscillating, where $k(x) \leq 0$ for each x(t).

Proof. It follows from Theorem 1 that (1) does not admit a correct non-negative strongly non-oscillating solution. Let us assume that there exists a correct non-positive strongly non-oscillating solution x(t), $x(t) \leq 0$ when $t \geq t^*$, $t^* \in I$. Then, by virtue of Lemma 2, $\liminf_{t \to +\infty} |x(t)| = 0$. On the other hand, it follows from the proof of Lemma 2 that $[(Lx)(t)]' \leq 0$ when $t \geq t^*$ and since x(t) is a correct solution, there exist a point $\overline{t} \geq t^*$ and a constant c > 0 such that $(Lx)(t) \leq -c < 0$ when $t \geq \overline{t}$.

Hence we obtain from Lemma 1

$$\operatorname{mess}\left(A = \left\{t \mid s \leq t \leq s + 2\tau, \ x(t - \tau) \leq -\beta < 0\right\}\right) \geq \tau$$

when $s \in [t, +\infty)$ and, if we put $k(x) = -\beta$, we conclude that the function $x(t) + \beta$ is strongly oscillating.

This proves Theorem 2.

References

- [1] Norkin S. В.: Осциляция решений дифференциальных уравнений с отклоняющимся аргументом. Дифференциальные уравнения с отклоняющимся аргументом, Наукова думка, Киев, 1977, 247—256.
- [2] A. I. Zahariev, D. D. Bainov: Oscillating properties of the solutions of a class of neutral type functional differential equations. Bull. Austral. Math. Soc. vol. 22 (1980), 365-372.

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