Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 2, 319–321

Persistent URL: http://dml.cz/dmlcz/101954

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

A NOTE ON THE STRONGLY COMPATIBLE TOLERANCES ON AN ARBITRARY SEMIGROUP

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(Received June 8, 1983)

In an earlier paper [3] the author proved that in a regular semigroup a strongly compatible tolerance is an idempotent separating congruence if it is contained in Green's relation $\mathscr{H}$. However, in an aribitrary semigroup it is not so. In this note we obtain sufficient conditions for a strongly compatible tolerance to be an idempotent separating congruence on a semigroup in general.

A reflexive and symmetric relation on a semigroup S is called a *tolerance on* S [4]. If ξ is a tolerance on S and if $(a, b), (c, d) \in \xi$ implies that $(ac, bd) \in \xi$ then ξ is called a *strongly compatible tolerance on* S. A non empty subset B of S is called a *block* [1] *of* ξ provided (i) $B \times B \subseteq \xi$ (ii) B is a maximal subset of S with respect to (i), that is, if $B \subseteq C$ and $C \times C \subseteq \xi$ then $B = C$.

For undefined terms and notions the reader is referred to [2].

Let ξ be a strongly compatible tolerance on a semigroup S. We observe that the condition "$\xi \subseteq \mathscr{H}$" is neither necessary nor sufficient for ξ to be an idempotent separating congruence on S.

This is illustrated by the following examples.

Example 1 [2]. Let $S = \{x, o\}$ be a null semigroup. Here $\mathscr{H} = 1_S$ and $S \times S$ is an idempotent separating congruence. Thus it is not necessary for any congruence, in particular a strongly compatible tolerance, to be contained in $\mathscr{H}$ in order that it should be an idempotent separating congruence.

Example 2. $S = \{a, b, c, d, e, f, g\}$ is a semigroup with the multiplication table given below:

	a	b	c	d	e	f	g
a	a	a	a	d	d	d	d
b	a	b	c	d	d	d	d
c	a	c	b	d	d	d	d
d	d	d	d	a	a	a	a
e	d	e	e	a	a	a	a
f	d	d	d	a	a	a	a
g	d	d	d	a	a	a	a

$\mathscr{H}$-classes of S are $\{a, d, f, g\}$, $\{b, c\}$ and $\{e\}$. Let ξ be a relation defined on S such that $\xi = \{(a, a), (a, d), (d, a)\ (d, d), (d, f), (d, g), (f, d), (f, f), (f, g), (g, d), (g, f), (g, g), (b, b), (c, c), (e, e)\}$. It is easy to see that ξ is contained in $\mathscr{H}$ and that it is a strongly compatible tolerance on S. However, ξ is not a congruence since (a, d), $(d, f) \in \xi$ but $(a, f) \notin \xi$. This shows that "$\xi \subseteq \mathscr{H}$" is not a sufficient condition for a strongly compatible tolerance to be an idempotent separating congruence on an arbitrary semigroup.

In the following theorem we obtain sufficient conditions for a strongly compatible tolerance to be an idempotent separating congruence on an arbitrary semigroup.

Theorem. *Let S be an arbitrary semigroup. A strongly compatible tolerance ξ on S is an idempotent separating congruence on S if* (i) $\xi \subseteq \mathscr{H}$ *and* (ii) *for the blocks* $\{B_i\}_{i \in I}$ *of ξ either $B_i \cap B_j = \emptyset$ $(i \neq j)$, or $B_i \cap B_j$ $(i \neq j)$ contains an idempotent.*

Proof. The conditions $\xi \subseteq \mathscr{H}$ and $B_i \cap B_j = \emptyset$ $(i \neq j)$ imply that the blocks form a partition of S and hence ξ is an idempotent separating congruence. Alternatively if $\xi \subseteq \mathscr{H}$ and $B_i \cap B_j$ contains an idempotent then H, the $\mathscr{H}$-class containing B_i and B_j, contains an idempotent and hence a subgroup of S ([2] Theorem 2.5).

If $a, b \in B_i$ and $b, c \in B_j$ we have $(a, b) \in \xi$ and $(b, c) \in \xi$. Now $(a, b) \in \xi$, $(\bar{b}, \bar{b}) \in \xi$ (where $\bar{b}$ is the group inverse of b in H) imply $(a\bar{b}, \bar{b}\bar{b}) \in \xi$; $(a\bar{b}, b\bar{b}) \in \xi$ and $(b, c) \in \xi$ together imply $(a\bar{b}b, b\bar{b}c) \in \xi$, that is $(a, c) \in \xi$, $\bar{b}b$ and $b\bar{b}$ being equal to the group identity in H. ξ is therefore a congruence on S. "$\xi \subseteq \mathscr{H}$" implies that ξ is an idempotential separating congruence on S.

Example 2 shows that condition (i) of the above theorem alone is not sufficient for a strongly compatible tolerance to be an idempotent separating congruence on an arbitrary semigroup.

The example which follows illustrates that condition (i) is an essential condition for a strongly compatible tolerance to be an idempotent separating congruence on an arbitrary semigroup. It further shows that condition (ii) of the theorem by itself is not sufficient for a strongly compatible tolerance on an arbitrary semigroup to be an idempotent separating congruence.

Example 3. Let $S = \{x, y, o\}$ be a null semigroup; clearly $\mathscr{H} = 1_S$ on S. Let a relation ξ on S be defined as $\xi = \{(x, x), (x, o), (o, x), (o, o), (y, y), (y, o), (o, y)\}$. Obviously ξ is a strongly compatible tolerance on S which is not contained in $\mathscr{H}$. Clearly (x, o), $(o, y) \in \xi$ but $(x, y) \notin \xi$. Thus ξ is not a congruence. However, the two blocks of ξ, i.e. $\{x, o\}$, $\{o, y\}$ have an idempotent in their intersection.

We note that in the case of regular semigroups condition (ii) is implied by condition (1) of the above theorem.

Acknowledgement. The author wishes to thank Dr. Vijaya L. Mannepalli for her helpful suggestions in the preparation of this paper and the referee for his valuable comments.

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